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ON THE THERMAL CONDUCTIVITIES OF CERTAIN POOR CONDUCTORS. — I.

By B. O. Peirce and R. W. Willson.

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ON THE THERMAL CONDUCTIVITIES OF CERTAIN POOR CONDUCTORS. — I.

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We have been engaged for several years in an attempt to measure, by the aid of the so called "Wall Method," the thermal conductivities of certain relatively poor conductors,* and the variations of these conductivities with the temperature. We have at length succeeded in overcoming some of the difficulties which we have encountered, and are now ready to describe our apparatus and to give the results of a number of observations made with it.

When one end of a regular right prism of $2n$ sides made of homogeneous material is kept at a constant temperature, V_0, and the other end at a constant temperature, V_1, while its other faces are kept as nearly as possible at some constant temperature between V_0 and V_1, the temperatures on the axis of the prism in its final state depend very largely on the ratio of the length of the axis of the prism to that of a diagonal of a cross

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* Despretz, Ann. de Chimie et de Physique, 1827. Peclet, Ann. de Chimie et de Physique, 1841. Tyndall, Phil. Mag., 1853. Hopkins, Phil. Trans., 1857. Pfaff, Pogg. Ann., CXIII., 1861. J. D. Forbes, Proc. Edin. Soc., IV. Ångström, Pogg. Ann., CXIV., 1861. Neumann, Ann. de Chimie et de Physique, 1862. G. Forbes, Proc. Edin. Soc., VIII., 1873. Herschell, Lebour, and Dunn, Rep. Brit. Assoc., 1873. v. Beetz, Pogg. Ann. Jubelband, 1874. Smith and Knott, Proc. Edin. Soc., 1875. Lodge, Phil. Mag. 1878. Less, Journ. de Phys., VII., 1878. Ayrton and Perry, Phil. Mag., 1878. H. F. Weber, Vierteljahrschrift d. Züricher Naturf. Ges., 1879. Thoulet, Comptes Rendus, 1882. Lagarde, Comptes Rendus, 1882. v. Littrow, Wien. Ber., LXXI. Stefan, Carl's Rep., XIII. Jannettaz, Comptes Rendus, 1884. Tuchschmid, Beiblätter z. Wied. Ann., 1884. M. Ballo, Dingler's Journ., 1886. H. Meyer, Wied. Ann., 1888. K. Jamagawa, Beiblätter z. Wied. Ann., 1889. G. Stadler, Inaug. Diss., Berne, 1889. Venske, Göttinger Nachrichten, 1891. Grassi, Atti Ist. Napoli, 1892. Lees, Phil. Trans., 1892. R. Weber, Bull. Soc. Science Nat. Neuch., 1895. Lord Kelvin and Mr. Murray, Proc. Royal Soc., 1895. Peirce and Willson, American Journal of Science, 1895. Lees and Chorlton, Phil. Mag., 1896. Oddone, Rend. R. Acc. d. Lincei, 1897. W. Voigt, Wied. Ann., 1898. Lees, Proc. Royal Society, 1898.

section; and, if this ratio be small enough, the temperature conditions to which the sides are subjected are of slight importance. For instance, the temperatures at points on the axis of a relatively thin disk, one face of which is kept at 0° C. and the other at 100° C., are not measurably different, whether the curved surface is kept at 0° C. or 100° C., from the temperatures at corresponding points on the axis of an infinite disk of the same thickness, the faces of which are kept at 0° C. and 100° C. respectively.

On the other hand, if the temperature gradient on the side faces could be made to follow the proper law, — or even if, for moderate values of $V_0 - V_b$, it could be kept constant, — the temperatures on the axis of the prism would be much the same, whether the prism were slender or stout.

In view of the extreme difficulty of controlling, or even of measuring with accuracy, the temperatures on the side faces of a prism, it seemed to us desirable to determine beforehand, as accurately as we could from theoretical considerations, under each of a number of different assumptions with respect to the side temperatures, how short a prism of given cross section must be in order that the temperatures on its axis, in the case mentioned above, might be sensibly the same as if its cross section were infinite in area.

We shall find it convenient to write down at the beginning of our discussion some of the common equations * of the theory of heat conduction in the forms which we shall need to use later on. If θ represents the temperature at the time t at any point, P, in an isotropic solid, the rate of flow of heat at this time, at P, in any direction, is usually assumed to be the product of a scalar point function, κ', and the negative of the space derivative, taken at P in the given direction, of a certain function of the temperature, $f(\theta)$. If, therefore, u, v, and w are the components, parallel to three mutually perpendicular co-ordinate axes, of the vector, q, which represents the flow within the solid,

$$u = -\kappa' \cdot \frac{\partial f(\theta)}{\partial x} = -\kappa' \cdot f'(\theta) \cdot \frac{\partial \theta}{\partial x},$$

$$v = -\kappa' \cdot f'(\theta) \cdot \frac{\partial \theta}{\partial y}, \qquad w = -\kappa' \cdot f'(\theta) \cdot \frac{\partial \theta}{\partial z}. \tag{1}$$

* Fourier, Théorie Analytique de la Chaleur. Poisson, Théorie Mathématique de la Chaleur. Lamé, Leçons sur la Théorie Analytique de la Chaleur. Kelvin, Article "Heat" in the Encyclopædia Britannica. Kelland, Brit. Assoc. Rep., 1841. Preston, Theory of Heat. Riemann, Partielle Differentialgleichungen.

If ξ, η, ζ are analytic point functions which define a system of orthogonal curvilinear co-ordinates, and h_ξ, h_η, h_ζ are the gradients of these functions, and if q_ξ, q_η, q_ζ are the components of the heat flux taken at every point normal to the surfaces of constant ξ, η, ζ which pass through that point,

$$q_\xi = -\kappa' . h_\xi . \frac{\partial f(\theta)}{\partial \xi}, \quad q_\eta = -\kappa' . h_\eta . \frac{\partial f(\theta)}{\partial \eta}, \quad q_\zeta = -\kappa' . h_\zeta . \frac{\partial f(\theta)}{\partial \zeta}. \qquad (2)$$

For a given material which would be homogeneous if it were at the same temperature throughout, under given pressure conditions, κ' is assumed to be constant, so that $\kappa' f'(\theta)$ is a function of the temperature only. This product is called the specific conductivity of the substance under the given circumstances, and is denoted by $F'(\theta)$ or by κ. We may write, therefore,

$$u = -\kappa . \frac{\partial \theta}{\partial x} = -\frac{\partial F(\theta)}{\partial x}, \quad v = -\frac{\partial F(\theta)}{\partial y}, \quad w = -\frac{\partial F(\theta)}{\partial z}. \qquad (3)$$

$$q_\xi = -h_\xi . \frac{\partial F(\theta)}{\partial \xi}, \quad q_\eta = -h_\eta . \frac{\partial F(\theta)}{\partial \eta}, \quad q_\zeta = -h_\zeta . \frac{\partial F(\theta)}{\partial \zeta}. \qquad (4)$$

If a closed analytic surface, S, be drawn within the solid and if (ξ, n), (η, n), (ζ, n) represent the angles between the exterior normal to S at any point on it and the directions at that point in which ξ, η, and ζ increase most rapidly, the flux of heat across S from within outward may be written

$$\int \{ q_\xi . \cos(\xi, n) + q_\eta . \cos(\eta, n) + q_\zeta . \cos(\zeta, n) \} \, dS. \qquad (5)$$

The surface integral, taken over S, of $U \cos(\xi, n)$, where U is any function which, with its space derivatives of the first order, is continuous within and upon S, is equal to the volume integral, extended through the space enclosed by S, of $h_\xi . h_\eta . h_\zeta . \dfrac{\partial \left(\dfrac{U}{h_\eta h_\zeta} \right)}{\partial \xi}$, so that the flux across S may be expressed by the integral

$$\iiint h_\xi . h_\eta . h_\zeta \left\{ \frac{\partial \left(\dfrac{q_\xi}{h_\eta h_\zeta} \right)}{\partial \xi} + \frac{\partial \left(\dfrac{q_\eta}{h_\zeta h_\xi} \right)}{\partial \eta} + \frac{\partial \left(\dfrac{q_\zeta}{h_\xi h_\eta} \right)}{\partial \zeta} \right\} d\tau. \qquad (6)$$

If $\psi(\theta)$ is the specific heat per unit volume of the body under the given pressure conditions, we may equate the expression just obtained to

$\iiint \psi(\theta) \cdot \frac{\partial \theta}{\partial t} \cdot d\tau$, and, since this result is independent of the form of S and of the volume of the space enclosed by it, at every point within the solid

$$\frac{\partial \theta}{\partial t} = -\frac{h_\xi \cdot h_\eta \cdot h_\zeta}{\psi(\theta)} \left\{ \frac{\partial\left(\frac{q_\xi}{h_\eta h_\zeta}\right)}{\partial \xi} + \frac{\partial\left(\frac{q_\eta}{h_\zeta h_\xi}\right)}{\partial \eta} + \frac{\partial\left(\frac{q_\zeta}{h_\xi h_\eta}\right)}{\partial \zeta} \right\}, \tag{7}$$

or $$\frac{\partial \theta}{\partial t} = \frac{h_\xi \cdot h_\eta \cdot h_\zeta}{\psi(\theta)} \left\{ \frac{\partial}{\partial \xi}\left(\frac{\kappa' h_\xi}{h_\eta h_\zeta} \cdot \frac{\partial f(\theta)}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\kappa' h_\eta}{h_\xi h_\zeta} \cdot \frac{\partial f(\theta)}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\kappa' h_\zeta}{h_\xi h_\eta} \cdot \frac{\partial f(\theta)}{\partial \zeta}\right) \right\}, \tag{8}$$

or $$\frac{\partial \theta}{\partial t} = \frac{h_\xi \cdot h_\eta \cdot h_\zeta}{\psi(\theta)} \left\{ \frac{\partial}{\partial \xi}\left(\frac{h_\xi}{h_\eta h_\zeta} \cdot \frac{\partial F(\theta)}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_\eta}{h_\xi h_\zeta} \cdot \frac{\partial F(\theta)}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{h_\zeta}{h_\xi h_\eta} \cdot \frac{\partial F(\theta)}{\partial \zeta}\right) \right\}, \tag{9}$$

three different forms of the equation of continuity.

In Cartesian co-ordinates, this equation becomes

$$\frac{\partial \theta}{\partial t} = \frac{1}{\psi(\theta)} \left\{ \frac{\partial}{\partial x}\left(\frac{\kappa' \cdot \partial f(\theta)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\kappa' \cdot \partial f(\theta)}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\kappa' \, \partial f(\theta)}{\partial z}\right) \right\}$$
$$\equiv \frac{1}{\psi(\theta)} \cdot \nabla^2 F(\theta). \tag{10}$$

$$= \frac{1}{\psi(\theta)} \left\{ \frac{\partial}{\partial x}\left(F'(\theta) \cdot \frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(F'(\theta) \cdot \frac{\partial \theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(F'(\theta) \cdot \frac{\partial \theta}{\partial z}\right) \right\}. \tag{11}$$

If the flow of heat within a solid is steady, $\frac{\partial \theta}{\partial t}$ vanishes at every point, q is a solenoidal vector, and the equation of continuity in terms of Cartesian co-ordinates becomes

$$\frac{\partial}{\partial x}\left(\frac{\kappa' \cdot \partial f(\theta)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\kappa' \cdot \partial f(\theta)}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\kappa' \cdot \partial f(\theta)}{\partial z}\right) = 0. \tag{12}$$

Or

$$\frac{\partial}{\partial x}\left(F'(\theta) \cdot \frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(F'(\theta) \cdot \frac{\partial \theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(F'(\theta) \cdot \frac{\partial \theta}{\partial z}\right)$$
$$\equiv \nabla^2 F(\theta) = 0. \tag{13}$$

It is usually assumed that θ is continuous at the surface of separation of two isotropic solids of different conductivities. If n_1 and n_2 are normals at a point of such a surface drawn into the first and second conductors respectively, and if the flow of heat is steady,

$$\frac{\partial F_1(\theta)}{\partial n_1} + \frac{\partial F_2(\theta)}{\partial n_2} = 0, \quad \text{or} \quad \kappa'_1 \frac{\partial f_1(\theta)}{\partial n_1} + \kappa'_2 \frac{\partial f_2(\theta)}{\partial n_2} = 0,$$

$$\text{or} \quad \kappa_1 \frac{\partial \theta}{\partial n_1} + \kappa_2 \frac{\partial \theta}{\partial n_2} = 0. \tag{14}$$

If the temperature differences within a body are comparatively slight, we may often use Fourier's assumption and represent $f(\theta)$ approximately by θ itself. As we shall need to compare the solutions of certain simple problems in the steady flow of heat obtained on this hypothesis with the corresponding solutions obtained on the assumption that $f(\theta)$ and θ are not identical, we may note certain facts in passing. It is easy to prove by an elementary application of Green's Theorem that a function, V, which is harmonic within a given closed surface S, and which upon two given portions, S_1 and S_2, of S has the constant values C_1 and C_2 respectively, while at every other part of S its normal derivative is zero, is determined by these conditions. If this function has been found, it is easy to write down the unique function

$$V' \equiv \frac{C'_1 - C'_2}{C_1 - C_2} V + \frac{C'_2 C_1 - C_2 C'_1}{C_1 - C_2}, \tag{15}$$

which is harmonic within S, has the constant value C'_1 on S_1 and the constant value C'_2 on S_2, and the normal derivative of which vanishes at all points of S which do not belong to S_1 or S_2. The families of surfaces defined by the equations, $V =$ constant, $V' =$ constant, are identical. If, therefore, two given portions of the surface of a solid isotropic conductor in which there is a steady flow of heat be kept at constant temperatures (C_1 and C_2) while there is no flow across the rest of its surface, the function V, which on Fourier's hypothesis gives the temperatures at all points within the solid, is connected with the function V', which gives $f(\theta)$ on the assumption that this is not identical with θ itself, by means of the equation

$$V' \equiv \frac{f(C_1) - f(C_2)}{C_1 - C_2} V + \frac{C_1 . f(C_2) - C_2 . f(C_1)}{C_1 - C_2}, \tag{16}$$

and the forms of the isothermal surfaces are independent of the form of the function f.

Two harmonic functions can only have the same level surfaces when one is a linear function of the other. If upon n given portions, $S_1, S_2, S_3, \dots S_n$, of a given closed surface, S, V has the constant values $C_1, C_1, C_3, \dots C_n$, respectively, and V' the values, $F(C_1)$, $F(C_2)$, $F(C_3)$, . . . $F(C_n)$, while upon the remainder of S, if there is any, the normal derivatives of V and V' are zero, and if V and V' are harmonic within S, V' cannot in general be expressed as a linear function of V, and, if n is greater than 2, their level surfaces will not usually coincide. If n is 3, the condition of coincidence is evidently

$$\begin{vmatrix} C_1 & F(C_1) & 1 \\ C_2 & F(C_2) & 1 \\ C_3 & F(C_3) & 1 \end{vmatrix} = 0. \tag{17}$$

If U has the constant values C_1, C_2, C_3; V the constant values K_1, K_2, K_3; and W the constant values L_1, L_2, L_3 on S_1, S_2, S_3, respectively, if the normal derivatives of these functions are equal to zero at every point of S not included in S_1, S_2, or S_3, and if all these functions are harmonic within S, W can always be expressed uniquely in the form $AU + BV + D$, unless

$$\begin{vmatrix} C_1 & K_1 & 1 \\ C_2 & K_2 & 1 \\ C_3 & K_3 & 1 \end{vmatrix} = 0. \tag{18}$$

Before we were able to decide upon the forms and dimensions of our apparatus and upon the manner in which it should be used, we found it desirable to make some rather elaborate computations based on the mathematical solutions of certain problems in heat conduction. In describing this work it will be convenient to state, first, some analytical results to which we shall afterwards give various physical interpretations. We have purposely put these preliminary statements in purely mathematical language lest they should seem to be narrower in their applications than they really are.

(1) The square bases of a rectangular parallelopiped of height l are $2a$ long and $2a$ broad. A function, V, harmonic within this parallelopiped, has the constant value V_0 at the lower base and the constant value V_l at the upper base. At every point of the other faces of the prism V satisfies the equation

$$\kappa \frac{\partial V}{\partial n} + h(V - \bar{V}) = 0, \tag{19}$$

where $\overline{V}$ is a constant, and $\frac{\partial V}{\partial n}$ represents the derivative of V taken in the direction of the exterior normal. If the origin of rectangular co-ordinates be taken at the centre of the lower base while the axes of x and y are parallel to the sides of this base, V is given by the equation

$$V \equiv \overline{V} + \sum_{p=1}^{p=\infty} c_p \cdot \cos(n_p y) \sum_{k=1}^{k=\infty} c_k \cdot \cos(n_k x)\ \Omega, \tag{20}$$

where Ω represents the quantity

$$\left[\frac{[(V_l-\overline{V})-(V_0-\overline{V})\epsilon^{-l\lambda_{p,k}}]\epsilon^{z\lambda_{p,k}}+[(V_0-\overline{V})\epsilon^{l\lambda_{p,k}}-(V_l-\overline{V})]\epsilon^{-z\lambda_{p,k}}}{\epsilon^{l\lambda_{p,k}}-\epsilon^{-l\lambda_{p,k}}}\right].$$

Here n_1, n_2, n_3, etc. are the successive roots of the equation

$$\kappa n \cdot \tan(n a) = h,$$

and $\lambda_{p,k}$ stands for the radical $\sqrt{n_p^2 + n_k^2}$, while c_1, c_2, c_3, etc. are the coefficients of the successive terms in the development,

$$1 = c_1 \cos(n_1\theta) + c_2 \cos(n_2\theta) + c_3 \cos(n_3\theta) + \ldots\ldots$$

so that $$c_k \equiv 4 \sin(n_k a) \div (2\, n_k a + \sin(2\, n_k a)).$$

It is to be noticed that equation (20) would give, on Fourier's assumptions, the final temperatures within a homogeneous parallelopiped of specific internal conductivity κ, and of external conductivity h, if the lower base were kept at the constant temperature V_0 and the upper base at the constant temperature V_l, while the sides were exposed to the atmosphere at the temperature $\overline{V}$. In this result the absolute dimensions of the parallelopiped are inextricably involved with the value of h / κ.

(2) The square bases of a rectangular parallelopiped of height l are $2a$ long and $2a$ broad. A function, V, harmonic within this parallelopiped, has the constant value V_0 at the lower square base, the constant value V_l at the upper base, and the constant value $\overline{V}$ on the other faces of the parallelopiped. If, then, the centre of the lower base be used as origin of co-ordinates, with axes of x and y parallel to sides of the base, V is given by the equation

$$V \equiv \overline{V} + \sum_{p=1}^{p=\infty} \sum_{q=1}^{q=\infty} (-1)^{\frac{p+q-2}{2}} \cdot \frac{4^2}{p\,q\,\pi^2} \cdot \cos\left(\frac{p\,\pi\,x}{2\,a}\right) \cdot \cos\left(\frac{q\,\pi\,y}{2\,a}\right) \cdot \Phi, \tag{21}$$

where Φ represents the quantity

$$\left\{\frac{(V_l-\overline{V})\sinh\left(\frac{\pi z}{2a}\sqrt{p^2+q^2}\right)-(V_0-\overline{V})\sinh\left(\frac{\pi(z-l)\sqrt{p^2+q^2}}{2a}\right)}{\sinh\left(\frac{l\pi}{2a}\sqrt{p^2+q^2}\right)}\right\},$$

and where p and q are integers.

V is evidently the temperature on Fourier's hypothesis within the parallelopiped, if its bases and sides are kept at the temperatures V_0, V_l, and $\overline{V}$ respectively, when the flow is steady. In this case the specific conductivity of the material of which the homogeneous parallelopiped is made does not affect the temperatures within the solid, and the relative, not the absolute, dimensions of the parallelopiped are of importance. The interpretation of the equation (21) when $f(\theta)$ and θ are assumed to be different is obvious.

(3) A function V, which involves the time and the distance from the co-ordinate plane $z=0$, is continuous, as are $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial z}$, $\frac{\partial^2 V}{\partial z^2}$, in the region R, bounded by the planes $z=0$, $z=l$. Within R, V satisfies the equation $\frac{\partial V}{\partial t}=a^2\frac{\partial^2 V}{\partial z^2}$. V vanishes when $z=l$, and has the constant value V_0 when $z=0$, whatever t is. If, when $t=0$, $V=V_0\phi(z)$ for all points within R,

$$V=V_0\left[1-\frac{z}{l}+\frac{2}{l}\sum_{m=0}^{m=\infty}e^{-\frac{m^2a^2\pi^2t}{l^2}}\sin\left(\frac{m\pi z}{l}\right)\cdot\right.$$
$$\left.\int_0^l[\psi(\lambda)+\frac{\lambda}{l}-1]\sin\frac{m\pi\lambda}{l}\,d\lambda\right]. \qquad (22)$$

If $\phi(z)$ has the constant value c,

$$V=V_0\left[1-\frac{z}{l}+\frac{2}{\pi}\left\{(2c-1)\left[e^{-\frac{t}{T}}\sin\frac{\pi z}{l}+\tfrac{1}{3}e^{-\frac{9t}{T}}\sin\frac{3\pi z}{l}+\right.\right.\right.$$
$$\left.\left.\left.\tfrac{1}{5}e^{-\frac{25t}{T}}\sin\frac{5\pi z}{l}+\ldots\right]-\left[\tfrac{1}{2}e^{-\frac{4t}{T}}\sin\frac{2\pi z}{l}+\tfrac{1}{4}e^{-\frac{16t}{T}}\sin\frac{4\pi z}{l}+\ldots\right]\right\}\right], \qquad (23)$$

where $T=l^2/a^2\pi^2$.

Equation (23) would give, on Fourier's assumptions, the temperatures at any time within a homogeneous infinite plane lamina of thickness l initially at the uniform temperature cV_0, if, from the time $t=0$, one face were kept at the constant temperature V_0 and the other at the constant temperature zero.

(4) The radius of the base of a right cylinder of revolution of length l is a. A function, V, harmonic within this cylinder, has the constant value V_0 on one (the lower) base, the constant value V_l on the upper base, and the constant value $\overline{V}$ on the convex surface. If, then, the axis of the cylinder be used as axis of z with origin at the centre of the lower base, V is given by the equation

$$V = V + 2 \sum_{p=1}^{p=\infty} \frac{J_0\left(\frac{x_p r}{a}\right)\left\{(V_0 - \overline{V}) \sinh\left(\frac{x_p(l-z)}{a}\right) + (V_l - V) \sinh\left(\frac{x_p z}{a}\right)\right\}}{x_p \,.\, J_1(x_p) \,.\, \sinh\left(\frac{x_p \,.\, l}{a}\right)}, \qquad (24)$$

where J_0 and J_1 represent Bessel's Function of the zeroth and first order, respectively, and x_p is the pth root in order of magnitude of the equation $J_0(x) = 0$. The first ten values of x for which the Bessel's Function of the zeroth order vanishes have been given by Meissel.* We have computed the next thirty values of the x_p's by the aid of Stokes's Formula.† and the values of the Bessel's Function of the first order corresponding to these forty x_p's either from the series which usually defines $J_1(x)$ or from the semi-convergent series. This computation was done by means of Vega's ten place table of logarithms,‡ except in the few cases where a greater number of places was necessary, and for these we had recourse to Thoman's tables.§ All the values have been checked by duplicate computation, and the first four values of $J_1(x)$ by comparison with Meissel's tables. The results of this work appear in Table I. Table II. gives to seven places of decimals the values of the x_p's from $p = 41$ to $p = 65$. The values of V on the axis of the cylinder depend upon the corresponding values of the function

$$S \equiv \sum_{p=1}^{p=\infty} \left\{ \frac{\sinh\left(\frac{x_p z}{a}\right)}{x_p \,.\, J(x_p) \sinh\left(\frac{x_p l}{a}\right)} \right\},$$

* Meissel, Math. Abhandlungen der k. Akad. der Wissenschaften zu Berlin, 1888.

† Stokes, Camb. Phil. Trans., IX. Lommel, Studien über die Bessel'schen Functionen, Leipzig, 1868. Rayleigh, The Theory of Sound, London, 1878. Byerly, Treatise on Fourier's Series, etc., Boston, 1893. Gray and Mathews, Bessel Functions and their Applications to Physics.

‡ Vega, Thesaurus Logarithmorum Completus, Lipsiæ, 1794.

§ Thoman, Tables de Logarithmes à 27 Décimales pour les Calculs de Précision, Paris, 1867.

TABLE I.

The pth *Root in Order of Magnitude of the Equation* $J_0(x) = 0$ *is denoted by* x_p.

p	x_p	Log x_p	$J_1(x_p)$	Log $(\pm J_1(x_p))$
1	2.4048255577	0.3810835788	+0.51914750	9.7152908
2	5.5200781103	0.7419452231	−0.34026481	9.5318170
3	8.6537279129	0.9372032361	+0.27145230	9.4336935
4	11.7915344391	1.0715703238	−0.23245983	9.3663479
5	14.9309177086	1.1740865018	+0.20654642	9.3150177
6	18.0710639679	1.2569837232	−0.18772880	9.2735309
7	21.2116366299	1.3265741787	+0.17326589	9.2387131
8	24.3524715308	1.3865430443	−0.16170155	9.2087142
9	27.4934791320	1.4392297006	+0.15218121	9.1823610
10	30.6346064684	1.4862123057	−0.14416598	9.1588628
11	33.7758202136	1.5286059043	+0.13729694	9.1376609
12	36.9170983537	1.5672275586	−0.13132463	9.1183462
13	40.0584257646	1.6026938781	+0.12606950	9.1006100
14	43.1997917132	1.6354816528	−0.12139863	9.0842138
15	46.3411883717	1.6659671666	+0.11721120	9.0689691
16	49.4826098974	1.6944525978	−0.11342918	9.0547248
17	52.6240518411	1.7211842839	+0.10999114$_5$	9.0413577
18	55.7655107550	1.7463656842	−0.10684789	9.0287659
19	58.9069839261	1.7701667872	+0.10395957	9.0168645
20	62.0484691902	1.7927310714	−0.10129350	9.0055816

TABLE I. — *Continued.*

The pth *Root in Order of Magnitude of the Equation* $J_0(x) = 0$ *is denoted by* x_p.

p	x_p	Log x_p	$J_1(x_p)$	Log $(\pm J_1(x_p))$
21	65.1899648002	1.8141807465	+0.09882255	8.9948561
22	68.3314693299	1.8346207594	−0.09652404	8.9846355
23	71.4729816036	1.8541418997	+0.09437879	8.9748744
24	74.6145006437	1.8728232368	−0.09237051	8.9655333
25	77.7560256304	1.8907340543	+0.09048519	8.9565775
26	80.8975558711	1.9079354006	−0.08871080	8.9479765
27	84.0390907769	1.9244813451	+0.08703686	8.9397032
28	87.1806298436	1.9404200023	−0.08545424	8.9317336
29	90.3221726372	1.9557943757	+0.08395493	8.9240462
30	93.4637187819	1.9706430570	−0.08253186	8.9166216
31	96.6052679510	1.9850008094	+0.08117879	8.9094426
32	99.7468198587	1.9988990584	−0.07989015	8.9024933
33	102.8883742542	2.0123663047	+0.07866100	8.8957595
34	106.0299309165	2.0254284784	−0.07748689	8.8892282
35	109.1714896498	2.0381625681	+0.07635913	8.8828610
36	112.3130502805	2.0504302219	−0.07528823	8.8767271
37	115.4546126537	2.0624112882	+0.07425684	8.8707365
38	118.5961766309	2.0740706879	−0.07326670	8.8649067
39	121.7377420880	2.0854252422	+0.07231515	8.8592293
40	124.8793089132	2.0964904866	−0.07139973	8.8536966

TABLE II.

The p*th Root in Order of Magnitude of the Equation* $J_0(x) = 0$ *is denoted by* x_p.

p	x_p	Log x_p	Log $\sqrt{\frac{2}{\pi x_p}}$
41	128.02087701	2.10728080	8.8482997
42	131.16244628	2.11780951	8.8430353
43	134.30401664	2.12808900	8.8378956
44	137.44558802	2.13823080	8.8328247
45	140.58716035	2.14794566	8.8279672
46	143.72873357	2.15754360	8.8231683
47	146.87030763	2.16693400	8.8184731
48	150.01188246	2.17617471	8.8138527
49	153.15345802	2.18512681	8.8093767
50	156.29503427	2.19394518	8.8049675
51	159.43601116	2.20258642	8.8006469
52	162.57818867	2.21106228	8.7964089
53	165.71976675	2.21937431	8.7922529
54	168.86134537	2.22753025	8.7881749
55	172.00292450	2.23553583	8.7841721
56	175.14450412	2.24339651	8.7802418
57	178.28608520	2.25111745	8.7763813
58	181.42766471	2.25870351	8.7725883
59	184.56924564	2.26615934	8.7688604
60	187.71082696	2.27348932	8.7651954
61	190.85240865	2.28069765	8.7615912
62	193.99399070	2.28778828	8.7580459
63	197.13557308	2.29476500	8.7545576
64	200.27715580	2.30163142	8.7511244
65	203.41873881	2.30838096	8.7477496

and these latter we have computed for certain values of z / l and a / l by the help of Gudermann's tables.* The results appear in Table III. To avoid possible errors arising from combining so many quantities, we generally used seven places, although the time required for the computation, which was done in duplicate, was thereby increased by some weeks.

* Gudermann, Theorie der Potenzial oder Cyklisch-hyperbolischen Functionen, Berlin, 1833. Willson and Peirce, Bulletin of the American Mathematical Society, 1897.

TABLE III.

$$S \equiv \sum_{0}^{\infty} \left\{ \frac{\sinh\left(\frac{x_p z}{a}\right)}{x_p J_1(x_p) \sinh\left(\frac{x_p l}{a}\right)} \right\}.$$

	$z = 0$	$z = \frac{1}{4} l$	$z = \frac{1}{2} l$	$z = \frac{3}{4} l$	$z = l$
$a = \frac{1}{4} l$	0	0.0006	0.0065	0.0703	0.5
$a = \frac{1}{2} l$	0	0.0196	0.0697	0.2116	0.5
$a = \frac{3}{4} l$	0	0.0558	0.1427	0.2908	0.5
$a = l$	0	0.0857	0.1920	0.3320	0.5
$a = \frac{3}{2} l$	0	0.1144	0.2349	0.3642	0.5
$a = 2 l$	0	0.1224	0.2464	0.3724	0.5
$a = 3 l$	0	0.1249	0.2498	0.3748	0.5
$a = 5 l$	0	0.1250	0.2500	0.3750	0.5

We shall wish to base an argument upon the values of S given in the last line of Table III., and upon certain corresponding values of the quantity

$$T \equiv \sum_{0}^{\infty} \frac{J_0\left(\frac{x_p r}{a}\right) . \sinh\left(\frac{x_p z}{a}\right)}{x_p . J_1(x_p) . \sinh\left(\frac{x_p l}{a}\right)}. \qquad (25)$$

We print, therefore, in Tables V. and VI., the numerical values of the terms of the series which define these functions in the cases in question.

It is evident that the three values of S are in reality less than 0.125, 0.250, 0.375, respectively, though by quantities far too small to appear in our results. Unavoidable errors introduced by adding together, in some instances, hundreds of numbers determined by logarithms, make the last figures given doubtful. Although our computations were made throughout with the aid of seven place and ten place tables, we have contented ourselves with four places in tabulating the values of T. It is interesting

to notice the seemingly anomalous sequence of values in the terms of the series for T. In fact, the relations between the successive terms is, for some cases that we have studied, so complicated that the detection of accidental errors of computation becomes extremely difficult. $T = 0$ when $r = a_1$ whatever z is, and equation (24) can be written in the form

$$V = \bar{V}(1 - 2\,T_z - 2\,T_{l-z}) + 2\,V_l\,T_z + 2\,V_0\,T_{l-z}.$$

TABLE IV.

a/l	z/l	$2\,S_z$	$1-2\,S_{l-z}$	$\frac{1}{2}+S_z-S_{l-z}$	$\frac{1}{2}-S_z-S_{l-z}$
$\frac{1}{4}$	$\frac{1}{4}$	.0012	.8595	.4303	.4291
$\frac{1}{4}$	$\frac{1}{2}$	.0130	.9870	.5000	.4870
$\frac{1}{4}$	$\frac{3}{4}$	.1405	.9988	.5695	.4291
$\frac{1}{2}$	$\frac{1}{4}$	.0395	.5768	.3080	.2688
$\frac{1}{2}$	$\frac{1}{2}$	.1393	.8607	.5000	.3606
$\frac{1}{2}$	$\frac{3}{4}$	.4232	.9607	.6920	.2688
$\frac{3}{4}$	$\frac{1}{4}$	.1117	.4185	.2651	.1534
$\frac{3}{4}$	$\frac{1}{2}$	.2854	.7146	.5000	.2146
$\frac{3}{4}$	$\frac{3}{4}$	.5815	.8883	.7349	.1534
1	$\frac{1}{4}$	.1714	.3359	.2536	.0823
1	$\frac{1}{2}$	.3839	.6161	.5000	.1160
1	$\frac{3}{4}$	.6641	.8286	.7463	.0823
$\frac{3}{2}$	$\frac{1}{4}$	.2288	.2716	.2502	.0214
$\frac{3}{2}$	$\frac{1}{2}$	.4698	.5302	.5000	.0302
$\frac{3}{2}$	$\frac{3}{4}$	.7284	.7712	.7498	.0214
2	$\frac{1}{4}$	.2449	.2552	.2501	.0052
2	$\frac{1}{2}$	.4927	.5073	.5000	.0062
2	$\frac{3}{4}$	.7448	.7551	.7499	.0052
3	$\frac{1}{4}$	.2497	.2503	.2500	.0003
3	$\frac{1}{2}$	.4996	.5004	.5000	.0004
3	$\frac{3}{4}$	.7497	.7503	.7500	.0003
5	$\frac{1}{4}$	.2500	.2500	.2500	.0000
5	$\frac{1}{2}$	.5000	.5000	.5000	.0000
5	$\frac{3}{4}$	.7500	.7500	.7500	.0000

TABLE V.

$$S \equiv \sum_{0}^{\infty} \left\{ \frac{\sinh\left(\frac{x_p z}{5l}\right)}{x_p \,.\, J_1(x_p) \sinh\left(\frac{x_p}{5}\right)} \right\}.$$

p	$z = \frac{1}{4} l$	$z = \frac{1}{2} l$	$z = \frac{3}{4} l$
1	+0.1931944	+0.389186	+0.590810
2	−0.1108619	−0.230224	−0.367236
3	+0.0694976	+0.152211	+0.263868
4	−0.0434740	−0.102502	−0.198205
5	+0.0268426	+0.069353	+0.152345
6	−0.0163951	−0.047111	−0.118978
7	+0.0099432	+0.032159	+0.094069
8	−0.0060054	−0.022070	−0.075105
9	+0.0036196	+0.015227	+0.060435
10	−0.0021801	−0.010557	−0.048940
11	+0.0013132	+0.007351	+0.039837
12	−0.0007914	−0.005139	−0.032567
13	+0.0004777	+0.003604	+0.026720
14	−0.0002886	−0.002536	−0.021990
15	+0.0001746	+0.001789	+0.018143
16	−0.0001057	−0.001264	−0.015007
17	+0.0000641	+0.000896	+0.012439
18	−0.0000389	−0.000635	−0.010325
19	+0.0000237	+0.000452	+0.008588
20	−0.0000144	−0.000321	−0.007150
21	+0.0000088	+0.000229	+0.005962
22	−0.0000053	−0.000163	−0.004975
23	+0.0000033	+0.000117	+0.004159
24	−0.0000020	−0.000083	−0.003479
25	+0.0000012	+0.000060	+0.002912
26	−0.0000007	−0.000043	−0.002441
27	+0.0000005	+0.000031	+0.002046
28	−0.0000002	−0.000022	−0.001717
29	+0.0000001	+0.000015	+0.001442
30	−0.0000001	−0.000011	−0.001211
31	. . .	+0.000008	+0.001018
32	. . .	−0.000006	−0.000856
33	. . .	+0.000004	+0.000721
34	. . .	−0.000003	−0.000607
35	. . .	+0.000002	+0.000511
36	. . .	−0.000002	−0.000430

TABLE V.— *Continued.*

p	$z = \frac{1}{4} l$	$z = \frac{1}{2} l$	$z = \frac{3}{4} l$
37	. . .	+0.000001	+0.000363
38	. . .	−0.000001	−0.000306
39	. . .	. . .	+0.000258
40	. . .	. . .	−0.000218
41	. . .	. . .	+0.000184
42	. . .	. . .	−0.000155
43	. . .	. . .	+0.000130
44	. . .	. . .	−0.000110
45	. . .	. . .	+0.000094
46	. . .	. . .	−0.000079
47	. . .	. . .	+0.000067
48	. . .	. . .	−0.000057
49	. . .	. . .	+0.000048
50	. . .	. . .	−0.000040
51	. . .	. . .	+0.000034
52	. . .	. . .	−0.000029
53	. . .	. . .	+0.000025
54	. . .	. . .	−0.000021
55	. . .	. . .	+0.000018
56	. . .	. . .	−0.000015
57	. . .	. . .	+0.000013
58	. . .	. . .	−0.000011
59	. . .	. . .	+0.000009
60	. . .	. . .	−0.000008
61	. . .	. . .	+0.000007
62	. . .	. . .	−0.000006
63	. . .	. . .	+0.000005
64	. . .	. . .	−0.000004
65	. . .	. . .	+0.000003
66	. . .	. . .	−0.000002
67	. . .	. . .	+0.000001
	0.1250000	0.250002	0.375004

TABLE VI.

$$T \equiv \sum_{0}^{\infty} \frac{J_0\left(\frac{x_p}{5}\right) \sinh\left(\frac{x_p z}{5l}\right)}{x_p \cdot J_1(x_p) \sinh\left(\frac{x_p}{5}\right)}.$$

p	$z = \frac{1}{4} l$	$z = \frac{1}{2} l$	$z = \frac{3}{4} l$
1	+0.182182	+0.367002	+0.557133
2	–0.079568	–0.165236	–0.263573
3	+0.026424	+0.057870	+0.100321
4	–0.001060	–0.002499	–0.004832
5	–0.006854	–0.017708	–0.038898
6	+0.006445	+0.018519	+0.046768
7	–0.003686	–0.011920	–0.034875
8	+0.001315	+0.004833	+0.016446
9	–0.000029	–0.000120	–0.000477
10	–0.000401	–0.001943	–0.009010
11	+0.000381	+0.002131	+0.011547
12	–0.000222	–0.001441	–0.009131
13	+0.000080	+0.000609	+0.004513
14	–0.000001	–0.000009	–0.000082
15	–0.000026	–0.000270	–0.002743
16	+0.000025	+0.000304	+0.003602
17	–0.000015	–0.000210	–0.002918
18	+0.000006	+0.000090	+0.001471
19	–0.000000	–0.000001	–0.000020
20	–0.000002	–0.000042	–0.000939
21	+0.000002	+0.000048	+0.001249
22	–0.000001	–0.000034	–0.001024
23	+0.000000	+0.000012	+0.000522
24	. . .	–0.000000	–0.000006
25	. . .	–0.000007	–0.000342
26	. . .	+0.000008	+0.000456
27	. . .	–0.000006	–0.000382
28	. . .	+0.000003	+0.000194
29	. . .	–0.000000	–0.000001
30	. . .	–0.000001	–0.000133
31	. . .	+0.000001	+0.000176
32	. . .	–0.000001	–0.000146
33	. . .	+0.000000	+0.000075
34	. . .	–0.000000	–0.000001
35	. . .	. . .	–0.000051
36	. . .	. . .	+0.000069

TABLE VI. — *Continued.*

p	$z=\frac{1}{4}l$	$z=\frac{1}{2}l$	$z=\frac{3}{4}l$
37	. . .	. . .	−0.000057
38	. . .	. . .	+0.000030
39	. . .	. . .	−0.000000
40	. . .	. . .	−0.000020
41	. . .	. . .	+0.000028
42	. . .	. . .	−0.000023
43	. . .	. . .	+0.000012
44	. . .	. . .	−0.000000
45	. . .	. . .	−0.000008
46	. . .	. . .	+0.000011
47	. . .	. . .	−0.000009
48	. . .	. . .	+0.000005
49	. . .	. . .	−0.000000
50	. . .	. . .	−0.000003
51	. . .	. . .	+0.000005
52	. . .	. . .	−0.000004
53	. . .	. . .	+0.000002
54	. . .	. . .	−0.000000
55	. . .	. . .	−0.000001
56	. . .	. . .	+0.000002
57	. . .	. . .	−0.000002
58	. . .	. . .	+0.000001
59	. . .	. . .	−0.000000
60	. . .	. . .	−0.000000
	+0.1250−	+0.2500−	+0.3749+

(5) The radius of the base of a right cylinder of revolution of height l is a. The centre of the lower base is used as the origin of a system of columnar co-ordinates (r, θ, z), the axis of the cylinder being the axis of z. A function V, which is continuous everywhere within the cylinder, has the value zero on the curved surface and on the lower base, and the constant value V_l on the upper base. The planes $z = l'$, $z = l''$, divide the cylinder into three portions (1), (2), and (3), in which V is represented analytically by three functions, V_1, V_2, V_3, respectively. If, when $z = l'$, $k_1 \dfrac{\partial V_1}{\partial z} = k_2 \dfrac{\partial V_2}{\partial z}$, and when $z = l''$, $k_2 \dfrac{\partial V_2}{\partial z} = k_3 \dfrac{\partial V_3}{\partial z}$, where k_1, k_2, k_3 are given constants, V_1, V_2, V_3 are given by the equations

$$V_1 \equiv \sum_{p=0}^{p=\infty} A_1 \,.\, J_0\left(\frac{x_p\, r}{a}\right) . \sinh\left(\frac{x_p\, z}{a}\right), \tag{26}$$

$$V_2 \equiv \sum_{p=0}^{p=\infty} J_0\left(\frac{x_p\, r}{a}\right)\left[A_2 \sinh\left(\frac{x_p\, z}{a}\right) + B_2 \cosh\left(\frac{x_p\, z}{a}\right)\right], \tag{27}$$

$$V_3 \equiv \sum_{p=0}^{p=\infty} J_0\left(\frac{x_p\, r}{a}\right)\left[A_3 \sinh\left(\frac{x_p\, z}{a}\right) + B_3 \cosh\left(\frac{x_p\, z}{a}\right)\right], \tag{28}$$

where A_1, A_2, A_3, B_2, and B_3 are subject to the conditions

$$A_1 \sinh\left(\frac{x_p\, l'}{a}\right) = A_2 \sinh\left(\frac{x_p\, l'}{a}\right) + B_2 \cosh\left(\frac{x_p\, l'}{a}\right), \tag{29}$$

$$k_1\, A_1 \cosh\left(\frac{x_p\, l'}{a}\right) = k_2\left[A_2 \cosh\left(\frac{x_p\, l'}{a}\right) + B_2 \sinh\left(\frac{x_p\, l'}{a}\right)\right],$$

$$A_2 \sinh\left(\frac{x_p\, l''}{a}\right) + B_2 \cosh\left(\frac{x_p\, l''}{a}\right) = A_3 \sinh\left(\frac{x_p\, l''}{a}\right) + B_3 \cosh\left(\frac{x_p\, l''}{a}\right),$$

$$k_2\left[A_2 \cosh\left(\frac{x_p\, l''}{a}\right) + B_2 \sinh\left(\frac{x_p\, l''}{a}\right)\right]$$
$$= k_3\left[A_3 \cosh\left(\frac{x_p\, l''}{a}\right) + B_3 \sinh\left(\frac{x_p\, l''}{a}\right)\right],$$

$$A_3 \sinh\left(\frac{x_p\, l}{a}\right) + B_3 \cosh\left(\frac{x_p\, l}{a}\right) = \frac{2\, V_l}{x_p\, J_1\,(x_p)},$$

and where x_p is the pth root in order of magnitude of the Bessel's Equation $J_0\,(x) = 0$.

If, for brevity, we denote the quantities

$$\sinh\left(\frac{x_p \,.\, l}{a}\right),\quad \cosh\left(\frac{x_p \,.\, l}{a}\right),\quad \sinh\left(\frac{x_p \,.\, l'}{a}\right),\quad \cosh\left(\frac{x_p \,.\, l'}{a}\right),$$

$$\sinh\left(\frac{x_p \,.\, l''}{a}\right),\quad \cosh\left(\frac{x_p \,.\, l''}{a}\right),\quad \frac{2\, V_l}{x_p \,.\, J_1\,(x_2)},$$

by s, c, s', c', s'', c'', and Ω, respectively, these equations of condition may be written

$$A_1 s' = A_2 s' + B_2 c', \qquad k_1 A_1 c' = k_2 (A_2 c' + B_2 s'),$$
$$A_2 s'' + B_2 c'' = A_3 s'' + B_3 c'', \quad k_2 A_2 c'' + k_2 B_2 s'' = k_3 A_3 c'' + k_3 B_3 s'',$$
$$A_3 s + B_3 c = \Omega.$$

The determinant of the coefficients of the s's and c's may be reduced to the form

$$\frac{1}{c}\begin{vmatrix} c' s' (k_1 - k_2) & 0 & k_1 c'^2 - k_2 s'^2 \\ c s'' & s c'' - c s'' & c c'' \\ k_2 c c'' & k_3 (s s'' - c c'') & k_2 c s'' \end{vmatrix} \tag{30}$$

and

$$A_1 = \frac{-k_2 k_3 c\, \Omega}{c's'(k_1 - k_2)\{k_2 c s''(sc'' - cs'') + k_3 c c''(cc'' ss'')\} + (k_1 c'^2 - k_2 s'^2)\{cs'' k_3 (ss'' - cc'') + k_2 c c''(s''c - sc'')\}} \tag{31}$$

If in the special case where k_1 and k_3 are equal, we write $k_1 = \mu\, k_2 = k_3$, we get

$$A_1 = \frac{-\mu\, \Omega}{c's'(\mu - 1)\{s''(sc'' - cs'') + \mu c''(cc'' - ss'')\} + (\mu c'^2 - s'^2)\{\mu s''(ss'' - cc'') + c''(s''c - sc'')\}} \tag{32}$$

with corresponding values for the other coefficients.

We shall need at the outset only two or three applications of the foregoing theory. We may ask first what must be the relative dimensions of a homogeneous regular right prism, one end of which is kept at the uniform temperature θ_0 and the other end at the uniform temperature θ_l, while its other faces are kept at some uniform temperature $\bar{\theta}$, between θ_0 and θ_l, in order that the temperatures on the axis of the prism in the final state shall be sensibly the same whatever value $\bar{\theta}$ has. If, for instance, the difference between θ_0 and θ_l is 100° C., what must be the ratio of the radius a of the circumference inscribed in a right section of the prism to the height l of the prism, in order that the temperature of every point on the axis may be the same within less than 0°.01 C., whether $\bar{\theta}$ is equal to θ_0 or to θ_l? Since we need merely to find a lower limit for $a \div l$, we shall do well to substitute for the prism the inscribed right cylinder of revolution, and then apply the solution of Problem 4 given above. We are to find a function of r and z, harmonic for values of r between 0 and a and values of z between 0 and l, which (1) has the uniform value $F(\theta_0)$ when $z = 0$, whatever r is; (2) has the uniform value $F(\theta_l)$

when $z = l$, whatever r is; and (3) has the uniform value $F(\bar{\theta})$ when $r = a$, whatever z is. The value of this function $F(\theta)$ is evidently

$$F(\bar{\theta}) \cdot (1 - 2\,T_{l-z} - 2\,T_z) + 2\,F(\theta_0) \cdot T_{l-z} + 2\,F(\theta_l) \cdot T_z, \qquad (33)$$

or, for points on the axis,

$$F(\bar{\theta})\,\{1 - 2\,S_{l-z} - 2\,S_z\} + 2\,F(\theta_0) \cdot S_{l-z} + 2\,F(\theta_l) \cdot S_z; \qquad (34)$$

that is,

$$F(\theta) - F(\theta_0) = 2\,S_z[F(\theta_l) - F(\theta_0)] + [F(\bar{\theta}) - F(\theta_0)]\,(1 - 2\,S_{l-z} - 2\,S_z). \qquad (35)$$

In the case of an infinite lamina, where $\frac{z}{a} = 0,\ \frac{l}{a} = 0,$

$$F(\theta) = F(\theta_0) + \frac{z}{l}\,(F(\theta_l) - F(\theta_0)). \qquad (36)$$

The difference between the values, at any point, of $F(\theta)$ in the case of the infinite lamina and in the case $a = 5\,l$, is

$$[F(\theta_l) - F(\theta_0)]\left[2\,S_z - \frac{z}{l}\right] + [F(\bar{\theta}) - F(\theta_0)]\,[1 - 2\,S_{l-z} - 2\,S_z].$$

It is easy to prove that for given values of l and a, $1 - 2\,S_{l-z} - 2\,S_z$ has its greatest value when $z = \frac{1}{2}\,l$, and if $a \div l$ is as great as 5, it is clear from Table V. that neither $1 - 2\,S_{l-z} - 2\,S_z$ nor $\left(2\,S_z - \frac{z}{l}\right)$ can for any point of the axis be nearly so great as 0.00001, so that whatever $\bar{\theta}$ is, the value of $F(\theta)$ is surely equal, within less than one ten-thousandth part of the greater of the quantities $F(\theta_l) - F(\theta_0)$, $F(\bar{\theta}) - F(\theta_0)$, to the value which it would have at the same point on the axis if the disk were infinite. By exactly what amount the temperatures themselves would differ in the two cases cannot be stated unless we know something of the nature of the function F.

For certain substances, experiment seems to show that within wide limits $F(\theta)$ can be expressed as a linear function of θ, as Fourier assumed. In the case of any one of these substances we may say, for example, that the final temperature at a point on the axis of a disk the radius of which is at least five times its thickness, if one face is kept at 100° C. and the other at 0° C., cannot be changed by nearly so much as 0°.01 C. by altering the temperature of the edge of the disk from 0° C.

to 100° C. The effect of radiation or conduction from the edge is therefore of no consequence.

Most experimenters have been able to reproduce mathematically the results of their work on thermal conductivities by assuming that in every case the conductivity, κ, is a linear function of θ, say $\kappa' (1 + 2b\theta)$, where b is small (usually less than .003), so that $F(\theta) = C + \kappa' \theta (1 + b\theta)$. On this assumption the temperatures within an infinite disk would be given by the equation,

$$\kappa' \theta (1 + b\theta) = \frac{\kappa' z}{l} \{\theta_l (1 + b\theta_l) - \theta_0 (1 + b\theta_0)\} + \kappa' \theta_0 (1 + b\theta_0), \quad (37)$$

or $$b(\theta^2 - \theta_0^2) + (\theta - \theta_0) = \frac{z}{l} \{\theta_l - \theta_0 + b(\theta_l^2 - \theta_0^2)\}.$$

Except in instances where near certain temperatures some great chemical or physical changes take place in the materials concerned, experiment appears to show that κ always changes slowly with the temperature, and, whether or not we know the exact nature of the connection between the two, it is easy to get a superior limit for the effect on the final temperatures at points on the axis of such a disk as has just been described, of changes in the edge temperatures. Neither in our own experience nor in any published reports that have come to our notice have we found any substance in which the change of κ with θ is so rapid that in a disk, where $a \geqq 5l$, made of it, with its faces kept at 0° C. and 100° C. respectively, the final temperatures of points on the axis could be affected by nearly so much as 0°.01 C. by changing the edge temperature from 0° C. to 100° C. We are here concerned merely with the magnitude of a possible error, and in every case to which we need to apply our theory we shall be well within bounds if we assume that the error is not greater than twice the error which would be found if θ and $f(\theta)$ were identical, as Fourier assumed them to be. We have, therefore, tabulated for a numerical example the final temperatures computed on Fourier's hypothesis at several points on the axis of a disk of radius a and length l, when one face ($z = 0$) is kept at the uniform temperature 0° C. and the other face ($z = l$) at the uniform temperature 100° C. on two or three different assumptions with respect to the edge temperatures. If the face temperatures are θ_0 and θ_l, and if the temperature has the same value, $\bar{\theta}$, at all points of the edge, the final axial temperatures are given by the equation

$$\theta = \bar{\theta}(1 - 2S_z - 2S_{l-z}) + 2\theta_0 S_{l-z} + 2\theta_l S_z,$$

and from this expression, with the help of the numbers in the body of Table IV., many special problems can be solved with very little labor. The expression

$$A\,(1 - 2\,T_z - 2\,T_{l-z}) + 2\,B\,T_{l-z} + 2\,\theta_l\,.\,T_z + (\theta_0 - B)\left(1 - \frac{z}{l}\right)$$

gives the final temperatures in a homogeneous disk of radius a and height l, one face ($z = 0$) of which is kept at the uniform temperature θ_0, the other face ($z = l$) at the uniform temperature θ_l, and the rim at constant temperatures given by the law $A + (\theta_0 - B)\left(1 - \frac{z}{l}\right)$. From this we may see, that, with a very rude approximation to a uniform gradient in the temperatures of the edge of a disk of relatively large thickness, the final temperatures on the axis are sensibly the same as for an infinite disk of the same thickness.

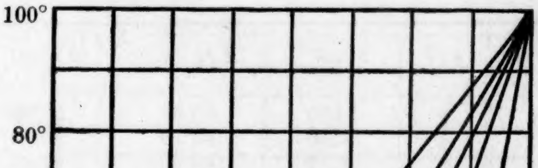

FIGURE 1.

Some of the results given in the first column of Table VII., with some others, are represented graphically in Figures 1 and 2. In Figure 1 the ordinates are the final axial temperatures; the abscissas, the distances from the cold face of the slab. The straight line corresponds to an infinite slab; the other curves, in order, to disks where $a = l$, $a = \frac{3}{4} l$, $a = \frac{1}{2} l$, and $a = \frac{1}{4} l$, respectively. In Figure 2, the ordinates of the three curves are the final temperatures on the axis at the points $z = \frac{l}{4}$, $z = \frac{l}{2}$, $z = \frac{3l}{4}$, respectively, and the abscissas are the values of a, each horizontal space corresponding to a change in a of $\frac{1}{2} l$.

TABLE VII.

Final Axial Temperatures in a homogeneous Disk of Radius a and Thickness l, when one Face ($z = 0$) is kept at 100° C., the other Face ($z = l$) at 0° C., and the Edge at the uniform Temperature $\bar{\theta}$.

a/l	z/l	$\bar{\theta} = 0°$	$\bar{\theta} = 100°$	$\bar{\theta} = 50°$
$\frac{1}{4}$	$\frac{1}{4}$	14.05	99.88	56.95
$\frac{1}{4}$	$\frac{1}{2}$	1.30	98.70	50.00
$\frac{1}{4}$	$\frac{3}{4}$	0.12	85.95	43.03
$\frac{1}{2}$	$\frac{1}{4}$	42.32	96.07	69.20
$\frac{1}{2}$	$\frac{1}{2}$	13.93	86.07	50.00
$\frac{1}{2}$	$\frac{3}{4}$	3.95	57.68	30.80
$\frac{3}{4}$	$\frac{1}{4}$	58.15	88.83	73.49
$\frac{3}{4}$	$\frac{1}{2}$	28.54	71.46	50.00
$\frac{3}{4}$	$\frac{3}{4}$	11.17	41.85	26.51
1	$\frac{1}{4}$	66.41	82.86	74.63
1	$\frac{1}{2}$	38.39	61.61	50.00
1	$\frac{3}{4}$	17.14	33.59	25.36
$\frac{3}{2}$	$\frac{1}{4}$	72.84	77.12	74.98
$\frac{3}{2}$	$\frac{1}{2}$	46.98	53.02	50.00
$\frac{3}{2}$	$\frac{3}{4}$	22.88	27.16	25.02
2	$\frac{1}{4}$	74.48	75.51	74.99
2	$\frac{1}{2}$	49.27	50.73	50.00
2	$\frac{3}{4}$	24.49	25.52	25.01
3	$\frac{1}{4}$	74.97	75.03	75.00
3	$\frac{1}{2}$	49.96	50.04	50.00
3	$\frac{3}{4}$	24.97	25.03	25.00
5	$\frac{1}{4}$	75.00	75.00	75.00
5	$\frac{1}{2}$	50.00	50.00	50.00
5	$\frac{3}{4}$	25.00	25.00	25.00

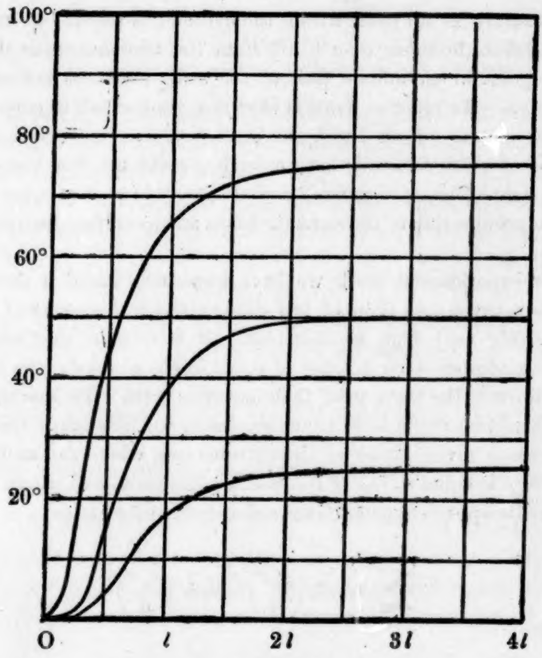

FIGURE 2.

If one is to measure the quantity of heat that passes through a portion of the disk, lying within a cylindrical surface of revolution of relatively small radius co-axial with the disk, it is desirable to make the ratio of a to l so large that possible changes in the edge temperatures shall not sensibly affect the temperatures at any point within the portion in question. It will be sufficient for our purpose to consider the temperatures at a distance l from the axis in a homogeneous disk for which $a = 5\,l$. It is evident that the greatest effect of temperature changes on the edge of the disk will apppear at those points on the inside cylindrical portion nearest the edge, that is, farthest from the axis.

Taking the formula

$$\theta = \bar{\theta}\,(1 - 2\,T_x - 2\,T_{l-x}) + 2\,\theta_0 \cdot T_{l-x} + 2\,\theta_l \,.\, T_x\,,$$

and using the values of T given in Table VI., we see that, if $\theta_0 = 100°$ C. and $\theta_l = 0°$ C., and, if the whole edge is kept at the temperature 0° C.,

the temperature at no point within the cylinder of radius l co-axial with the disk differs by more than 0°.02 from the temperature at the corresponding point in an infinite disk of the same thickness and same face temperatures. In practice there is always a gradual fall in edge temperatures as z increases from 0 to l, and in such a case we may consider a guard ring of width $4\,l$ amply large enough to make the final temperatures within a right cylinder of radius l and thickness l sensibly equal to those within an infinite slab of the same thickness and same face temperatures.

In our experimental work we have sometimes found it desirable to introduce between two slabs of low conductivity a thin sheet of tinfoil of comparatively very high conductivity. It is evident that under conceivable conditions such a layer of metal might seriously affect the final temperatures in the slabs near their common axis. To investigate the disturbances that might arise from this cause, we may apply the solution of Problem 5 given above to the extreme case where the uniform edge temperature is equal to one of the face temperatures, and where $k_1 = k_3$.

If we attempt to compute numerical values of the series

$$\sum_{p=0}^{p=\infty} A_1 \,.\, J_0\left(\frac{x_p\, r}{a}\right) \,.\, \sinh\left(\frac{x_p\, z}{a}\right)$$

by using the expression for A_1 given in equation (32), we shall find the amount of labor involved enormous; we will therefore change the form of the expression so as to make the nature of its dependence upon the dimensions of the cylinders and upon their conductivities more evident, keeping in mind the fact that the ratio of k_2 to k_1 is very large. If we denote the denominator of the second member of (32) by D, and write

$$\alpha \equiv \sinh^{-1} s', \qquad \alpha + \delta \equiv \sinh^{-1} s'', \qquad \text{and} \quad \alpha_0 \equiv \sinh^{-1} s\,,$$

we have

$$\begin{aligned} D = s\, &\{(\mu\, s''^2 - c''^2)(\mu\, c'^2 - s'^2) - (1-\mu)^2 c'\, s'\, c''\, s''\} \\ &\qquad + c\,(1-\mu)\{s''\, c''\,(\mu\, c'^2 - s'^2) - s'\, c'\,(\mu\, c''^2 - s''^2)\} \\ = \tfrac{1}{4} s &\{(1-\mu)^2 \cosh 2\delta + (\mu^2 - 1)[\cosh 2(\alpha+\delta) - \cosh 2\alpha] - (1+\mu)^2\} \\ &\qquad - \tfrac{1}{4} c\,(1-\mu)^2 \sinh 2\delta + \tfrac{1}{4} c\,(1-\mu^2)[\sinh 2(\alpha+\delta) - \sinh 2\alpha] \\ = \tfrac{1}{4} &\{(1-\mu)^2 \sinh(\alpha_0 - 2\delta) - (1+\mu)^2 \sinh \alpha_0 \\ &\qquad - (1-\mu^2)[\sinh(\alpha_0 - 2\alpha - 2\delta) - \sinh(\alpha_0 - 2\alpha)]\}, \end{aligned}$$

and $A_1 =$

$$\frac{4\,\mu\,\Omega}{(1+\mu)^2\sinh\alpha_0-(1-\mu)^2\sinh(\alpha_0-2\delta)+(1-\mu^2)[\sinh(\alpha_0-2\alpha-2\delta)-\sinh(\alpha_0-2\alpha)}.$$

This expression, though still sufficiently complicated, shows that for properly chosen cases, as good for our present purpose as any others, the computation is comparatively simple.

If, for instance, the thickness of the lower slab is half that of the disk formed of the two slabs and the intermediate sheet of metal, $l' = \frac{1}{2}\,l$ and $\alpha_0 = 2\,\alpha$, so that

$$A_1 = \frac{4\,\mu\,\Omega}{(1+\mu)^2\sinh\alpha_0-(1-\mu)^2\sinh(\alpha_0-2\,\delta)-(1-\mu^2)\sinh 2\,\delta} \quad (38)$$

$$= \frac{\Omega}{\sinh\alpha_0\left\{1-\frac{(1-\mu)^2}{2\,\mu}\sinh^2\delta+\frac{(\mu-1)\sinh^2\delta}{4\,\mu\sinh\alpha_0}[1-\cosh\alpha_0+\mu\,(1+\cosh\alpha_0)]\right\}}, \quad (39)$$

If we denote the denominator of this expression by $\sinh\alpha_0\,.\,(1+\Delta)$, and note that, if we make μ equal to unity, we shall have $A_1 = \frac{\Omega}{\sinh\alpha_0}$, corresponding to the case of a homogeneous cylinder already treated in Problem 4, we shall see that V_1 in the case of the heterogeneous cylinder can be found by multiplying each term of the series for T_s by the quantity $\frac{2\,V_l}{(1+\Delta)}$, and that in our problems the resulting series is usually more convergent than the original.

In order to exaggerate the magnitude of the disturbing effect of the tinfoil, we have chosen for computation a value of μ much smaller and a value of δ much greater than the proper values of these quantities for most of our experiments, assuming that $\alpha = 5\,l = 10\,l' = 500\,(l''-l')$, and that $\mu = 0.002$, so that Δ is nearly equal to

$$\tfrac{1}{2}\,x_p\tanh\tfrac{1}{2}\,\alpha_0 - \frac{x_p}{1000}\,\mathrm{ctnh}\,\tfrac{1}{2}\,\alpha_0 - \frac{x_p^{\,2}}{1000}, \quad \text{where} \quad \alpha_0 = \frac{x_p}{5}.$$

These values correspond in certain cases to large disturbances of temperature on the axis of the slabs, as the results show. Consider, for instance, the point $z = l'$, $r = 0$, in a compound slab 2 cm. thick and 20 cm. in diameter, built up of a slab of poorly conducting material 1 cm. thick, a sheet of metal 0.2 mm. thick, and a second slab of the same material as the first. Let the lower face be kept at the temperature 0° C., the other face at the temperature 100° C., and let every point of the edge be kept

at 0° C. The terms of the series which give the final temperature may be found without much difficulty by the aid of the numbers in the third column of Table V. Their values are

1.	+61.3980
2.	−19.6480
3.	+ 7.8876
4.	− 3.7116
5.	+ 1.8474
6.	− 1.0224
7.	+ 0.5938
8.	− 0.3568
9.	+ 0.2198
10.	− 0.1382
11.	+ 0.0882
12.	− 0.0570
13.	+ 0.0372
14.	− 0.0246
15.	+ 0.0162
16.	− 0.0110
17.	+ 0.0074
18.	− 0.0050
19.	+ 0.0034
20.	− 0.0022

etc., so that the temperature required is 47°.12+ C.

The terms of the series which give the final temperature at points for which $z = 1$, $r = 2$, can be found in a similar way by the help of the numbers in the third column of Table VI. Their values are

1.	+57.8982
2.	−14.1020
3.	+ 2.9988
4.	− 0.0904
5.	− 0.4716
6.	+ 0.4020
7.	− 0.2192
8.	+ 0.0782
9.	− 0.0018
10.	− 0.0254
11.	+ 0.0256

12.	$-$ 0.0160
13.	$+$ 0.0062
14.	$-$ 0.0000
15.	$-$ 0.0024
16.	$+$ 0.0026
17.	$-$ 0.0018
18.	$+$ 0.0008
19.	$-$ 0.0000
20.	$-$ 0.0002

and the temperature is 46°.48.

If the radii of the slabs and the metal sheet had been infinite, the temperature in these media would have been given by the expressions Mz, $M(\mu z + 1 - \mu)$, and $M(z + \frac{1}{50}(\mu - 1))$, respectively, where $M = 100 / (1.98 + .02\,\mu)$. In all practical cases the temperatures of points on the rim of the disk increase gradually from the cold face to the warm face, and it would be easy to show that those portions of the isothermal surfaces which we have used in computing the results of our observations are sensibly plane.

The characteristic differential equation which gives the relation between the temperature, the space co-ordinates, and the time in a body in which there is an unsteady flow of heat, involves the specific heat of the body, which is itself a function of the temperature. Without attempting just here to investigate the nearness of the approximation obtained in any given case by assuming the specific heat to be constant, we will give for future reference some numerical results obtained by using several different values of z, t, and c in the solution of Problem 3.

An infinite homogeneous lamina of thickness l is originally at the temperature $c\,V_0$ throughout. From a given time, $t = 0$, one face is kept at the constant temperature V_0, and the other face at the temperature 0°. The ratio of the conductivity of the slab to its specific heat is to be denoted by the constant a^2, the ratio of l^2 to $a^2\pi^2$ by T, and the distance of any point in the lamina from the face which is kept at the temperature V_0, by z.

The numbers in Table VIII. show the rate of flow across the cold face of the lamina in fractional parts of the final rate for different values of c and t, while the numbers in Table IX. give the rate of flow across different planes parallel to the lamina faces at different times, for the special case $c = \frac{1}{2}$.

TABLE VIII.

	$t=\frac{1}{8}T$	$t=\frac{1}{4}T$	$t=\frac{1}{2}T$	$t=T$	$t=2T$	$t=4T$	$t=6T$
$c=-1$	–5.013	–3.544	–2.434	–1.171	0.188	0.890	0.986
$c=-\frac{1}{2}$	–2.506	–1.772	–1.199	–0.435	0.459	0.927	0.990
$c=-\frac{1}{4}$	–1.253	–0.886	–0.582	–0.067	0.595	0.945	0.993
$c=0$	0.000	0.000	0.036	0.301	0.730	0.963	0.995
$c=\frac{1}{4}$	1.253	0.886	0.654	0.669	0.865	0.982	0.998
$c=\frac{1}{2}$	2.507	1.773	1.271	1.037	1.001	1.000	1.000
$c=1$	5.013	3.545	2.507	1.773	1.271	1.037	1.005

TABLE IX.

	$t=\frac{1}{8}T$	$t=\frac{1}{4}T$	$t=\frac{1}{2}T$	$t=T$	$t=2T$	$t=4T$	$t=6T$
$z=0$	3.760	2.659	1.889	1.405	1.136	1.018	1.002
$z=\frac{1}{8}l$	2.762	2.279	1.756	1.366	1.125	1.017	1.002
$z=\frac{1}{4}l$	1.095	1.438	1.420	1.260	1.096	1.013	1.002
$z=\frac{3}{8}l$	0.235	0.682	1.030	1.115	1.051	1.007	1.001
$z=\frac{1}{2}l$	0.036	0.301	0.730	0.963	0.999	1.000	1.000
$z=\frac{5}{8}l$	0.082	0.278	0.587	0.823	0.948	0.993	0.999
$z=\frac{3}{4}l$	0.365	0.489	0.578	0.740	0.904	0.987	0.998
$z=\frac{7}{8}l$	0.922	0.761	0.627	0.686	0.876	0.983	0.998
$z=l$	1.253	0.886	0.654	0.669	0.865	0.982	0.998

In Figure 3, the abscissas are the elapsed times (one division $= \frac{1}{2}\,T$), and the ordinate corresponding to any abscissa is the rate of flow of heat at that instant across every unit of surface of the cold face of the lamina

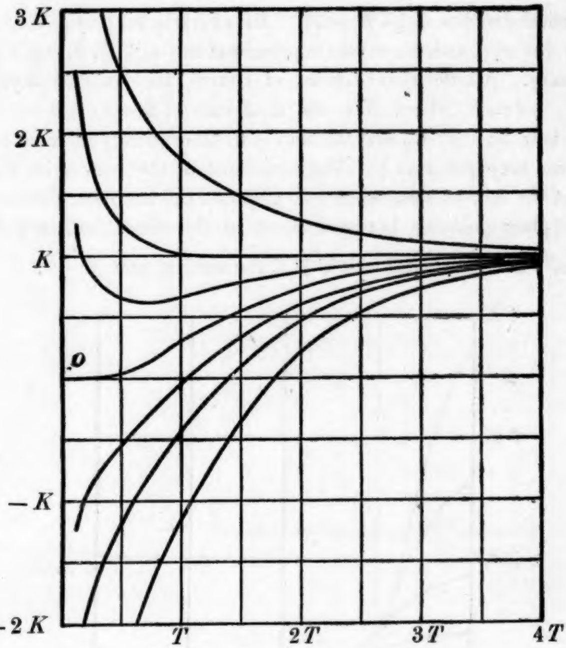

FIGURE 3.

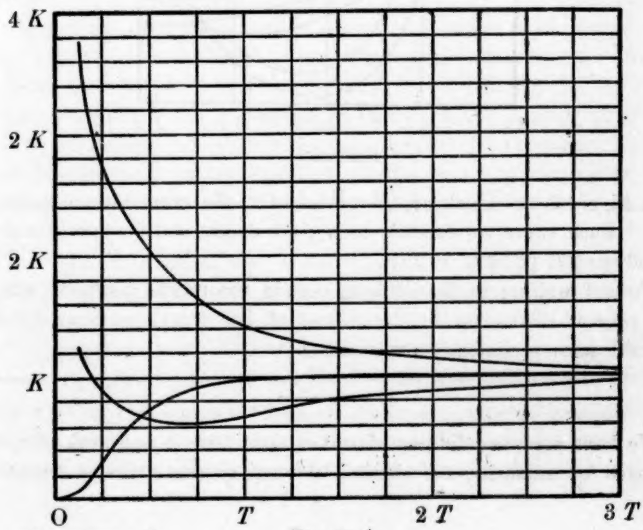

FIGURE 4.

(one vertical division $= \frac{1}{2} \kappa V_0 \div l$). Every curve corresponds to a particular value of c, and the values represented are 1, $\frac{1}{2}$, $\frac{1}{4}$, 0, $-\frac{1}{4}$, $-\frac{1}{2}$, -1, respectively. All the curves have, of course, the common asymptote, $y = \kappa V_0 \div l = K$, where K is the final rate of flow.

If V_0 is to be 100° C., and the slab is to be originally at room temperatures, we may put $c = \frac{1}{4}$. The ordinates of the curves in Figure 4 represent the flow of heat, when $c = \frac{1}{4}$, across the hot face, the cold face, and the plane midway between them, at the times indicated by the abscissas. The horizontal unit is $\frac{1}{4}$ T, the vertical unit $\frac{1}{8} \frac{k V_0}{l}$.

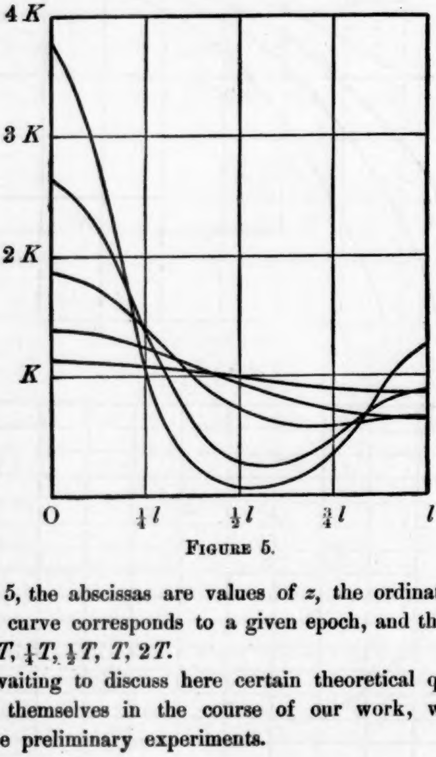

Figure 5.

In Figure 5, the abscissas are values of z, the ordinates are rates of flow. Each curve corresponds to a given epoch, and the epochs represented are $\frac{1}{8} T$, $\frac{1}{4} T$, $\frac{1}{2} T$, T, $2 T$.

Without waiting to discuss here certain theoretical questions which will present themselves in the course of our work, we may briefly describe some preliminary experiments.

We have used two different forms of apparatus in our work, the one intended for measuring the absolute thermal conductivities at tempera-

tures between 0° C. and 100° C. of relatively poor conductors like plates of stone or glass; the other designed merely for comparing the conductivities of slabs which form a prism or "wall," through which there is a steady flow of heat.

Of this second form of apparatus, which is much simpler than the other, we have three of different sizes for plates 65 cm., 35 cm., and 20 cm. in diameter respectively. Rough diagrams which show the essential parts of two of these, without their elaborate stands and jackets, are given in Figures 6 and 7. In each, the prism to be tested is enclosed

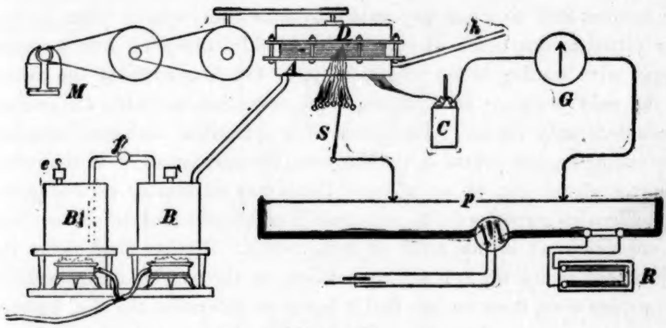

FIGURE 6.

between the horizontal planed plates of two castings, which are fastened firmly together by bolts around their edges to insure close contact with the body under experiment. Both castings are hollow; one forms a jacketed chamber through which steam or mercury vapor may be passed for an indefinite period. The upper casting, which is provided with a system of stirrers or scrapers operated by an electric motor, may be kept at a low temperature by filling it with ice or by sending through it a steady stream of water from a very large tank within the tower of the laboratory.

In Figure 6, A represents the hot chamber, weighing about two hundred kilograms, which rests in a thick jacket on a heavy table or stand made to hold it. A is connected directly with one (B) of two stout-walled copper boilers, B and B', each of which holds about 40 litres of water. A light cup-shaped weight, inverted and laid on a large tube with squared end which projects above the top of the boiler, acts as a sensitive safety valve and prevents any appreciable rise in temperature within the boiler. B can be refilled when necessary with boiling water

from *B′* without stopping the constant flow of steam through *A*, by means of the siphon *f*, which is provided with a valve. The steam, after passing through the hot chamber, is led to the outer air by a jacketed pipe *h*, descending * from the bottom of *A*.

The connections of the thermal elements are led out of the sides of the prism shut in by *A* and *D*, and are held between slabs of wood, which act as a sort of guard-ring jacket to the prism for about 40 centimeters before they emerge. The platinoid or German silver leads of these thermal junctions within the prism are soldered together, and to a copper wire leading to the (copper) wire of a potentiometer, *p*. The copper ends of the couples lead to a mercury switch by which any one of them, or any pair pitted against each other, may be quickly connected with a second copper wire leading to the potentiometer. On its way from the switch to the cold junctions in *C* through the potentiometer wire, the current encounters only copper. By means of a somewhat elaborate standard potentiometer, not shown in the diagram, the resistance, *R*, in the potentiometer circuit can be so adjusted that every millimeter on the potentiometer wire corresponds to any desired small potential difference, such as one microvolt or one tenth of a microvolt. Rather than make this adjustment many times a day to conform to the varying temperature of the copper wire, however, we find it better to determine the slight corrections necessary to reduce the readings to absolute measure, by noting at frequent intervals the indications of a standard thermal couple, the electromotive force of which is well known. The potentiometer wire, which is 0.25 mm. in diameter, can be changed in a few seconds for new wire, if the old should become dented or stretched.

Into the vessel *D* about 100 kilograms of cracked ice can be put, and this ice can be kept in constant motion over the smooth bottom by help of the electric motor, *M*.

Figure 7 shows a similar but smaller apparatus without its elaborate system of inch thick asbestos jackets. *D* is a closed iron drum containing a rotary stirrer and rubber scraper turned by a motor. Through *D* a large volume of water can be sent at a steady rate. The hot chamber is the iron box, *B*, planed on its upper surface and communicating at the bottom with a retort chamber, *C*, in which about 20 kilograms of mercury can be kept boiling. The outlet at *f* allows the vapor to escape to the tube *g*, connecting with a large wrought iron chamber where it condenses

* In the diagram, *h* is erroneously represented as *ascending*, and as inserted in the side of *A*.

and flows back into the retort through the trap h. This apparatus takes slabs 35 centimeters square. Although we found it possible to maintain with this arrangement a temperature above 350° C., for many hours at a time, it was difficult to avoid superheating by conduction through the massive iron of the hot box, and we intend to discard mercury in future and use some less troublesome source of heat. If a substance of greater heat of vaporization than mercury is employed, the retort can be removed to such a distance that all danger of superheating is removed. We have not yet been able to test an electrical stove which we hope may prove to be a convenient and a sufficiently constant source of heat for many purposes.

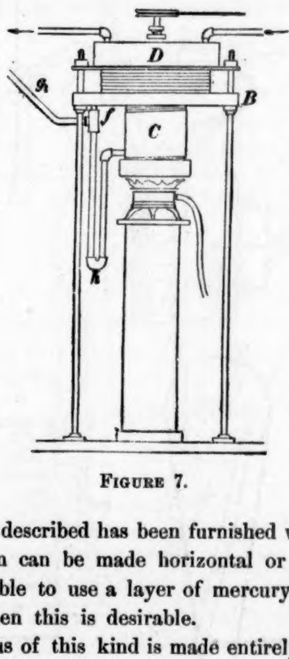

FIGURE 7.

The apparatus just described has been furnished with trunnions so that the axis of the prism can be made horizontal or vertical at pleasure. This renders it possible to use a layer of mercury on each side of the slab to be tested, when this is desirable.

Our third apparatus of this kind is made entirely of brass. It is intended only for small thin plates about 20 cm. in diameter, but is in essentials like the apparatus just described.

Figure 8 represents the apparatus which we have used to determine the absolute conductivities at temperatures between 0° C. and 100° C. of various materials. The boilers and the hot chamber are those of the apparatus shown in Figure 6; the ice box, which is the outcome of

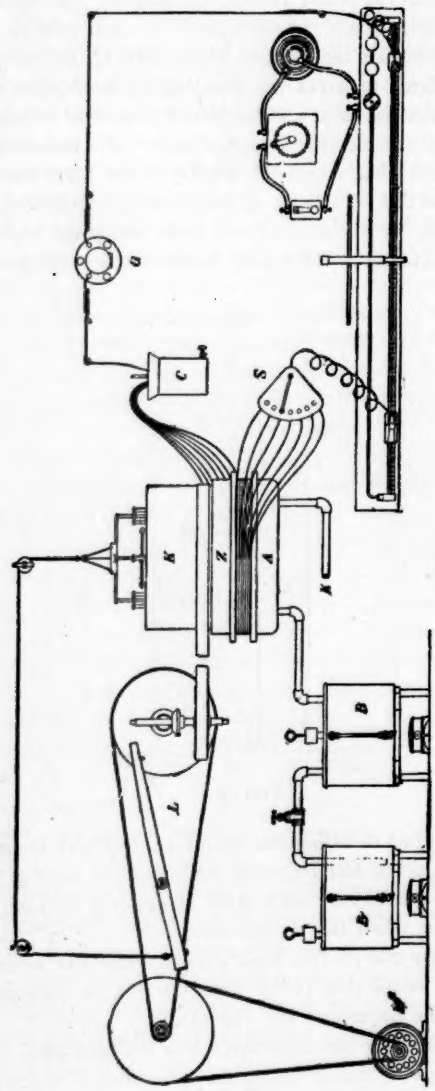

FIGURE 8.

several years of experimentation, is entirely different. An iron casting, *Z*, seen in plan in Figure 11 and in elevation in Figure 9, accurately planed below and turned true above, is the bottom of the box. Between this casting (which can be bolted to *A* as *D* is in Figure 6) and

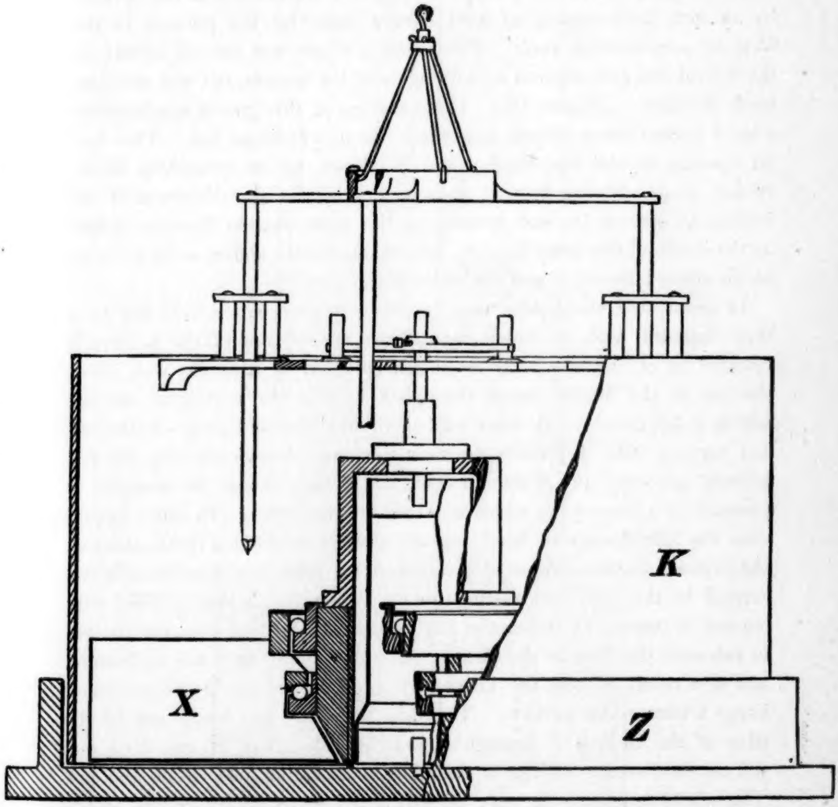

FIGURE 9.

A is held the prism to be experimented on. While *Z* was in the lathe a small hole, *H*, about 3 millimeters in diameter and 4 millimeters deep was drilled exactly in the centre of its upper face. Subsequently a piece of solid drawn brass tube 12.3 cm. in outside diameter and 13.5 cm. high, with carefully squared ends, was held centrally in *Z*, by means of a

wooden disk turned to fit it, and a central pin inserted in *H*, and was then soldered firmly to *Z*. This was accomplished, after many trials of other materials, by the use of white pitch as a flux, and the result left nothing to be desired. The walls of the pot thus formed were jacketed on the outside, except for a height of about 2 millimeters at the bottom, by an inch thick casting of hard rubber made for the purpose in the form of a cylindrical shell. This casting, which was cut off square at the top of the pot, tapered to nothing near the bottom, but did not rest upon the floor. (Figure 10.) Upon the top of this jacket was fastened a hard rubber cover shaped somewhat like a cylindrical hat. This had an opening at the top which could be closed by an accurately fitting rubber plug. In the box *P*, thus made, is placed a thin-walled ice holder, *Q*, open at top and bottom, of the same outside diameter below as the inside of the brass pot, but somewhat smaller above, so as to leave an air space between it and the walls of the pot.

In order that the holder may be easily rotated, a pin soldered to a thin diametral web, *F*, which runs across the bottom of the holder, is inserted in *H*, and a vertical brass rod soldered to a similar web, *E*, at the top of the holder passes through a hole in the corner of the pot which it fits closely. A hard rubber thimble fitting tightly on the rod and turning with it permits the slow entrance of cold air into the pot without allowing any water to leak in. The rod can be clamped at pleasure to a brass yoke which is turned by the motor. In order to prevent the introduction of heat into the pot by conduction down the rod, the exposed portion is buried in cracked ice held in a thin metallic cup carried by the yoke and resting on it. When the holder is filled with ice and is turned by the motor, the web at the bottom compels the ice to rub over the floor of the casting, since the holder itself has no bottom, and as a result of this, the lower surface of the ice quickly acquires and keeps a mirror-like surface. The drip from the pot comes out of the edge of the casting *Z* through a straight hole about 26 cm. long and 0.6 cm. in diameter drilled in the plate and ending just inside the pot. The whole apparatus is very slightly tilted so as to insure the steady outflow of the drip.

A large cylinder, *K*, 35 cm. high, made of rolled brass 4 mm. thick and open at the top and bottom, is mounted on brass ball bearings placed on the outside of the hard rubber jacket of the pot *P*, by means of six vanes, one of which, *X*, is shown in Figure 9. *K* weighs about 20 kilograms when empty, and rests upon 144 brass balls each about 12 mm. in diameter. When set in motion by a slight push, it continues to rotate

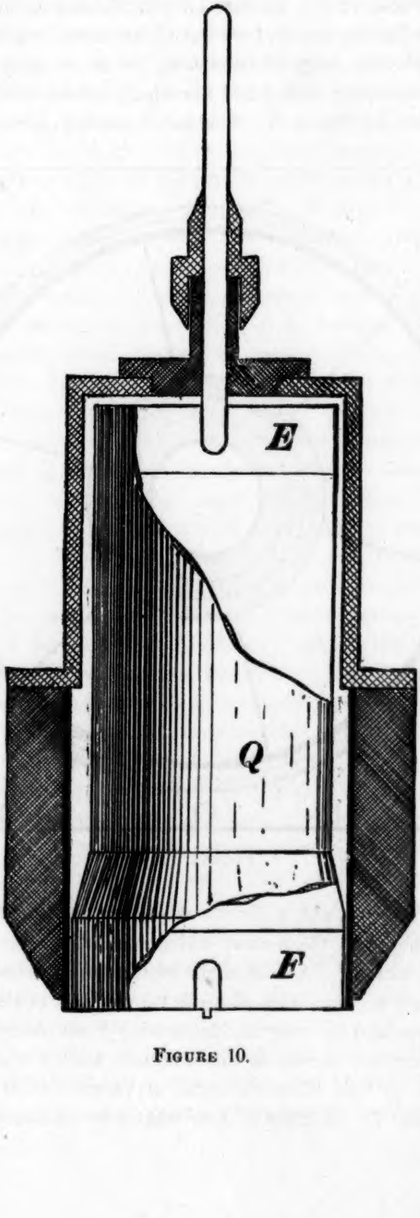

FIGURE 10.

for about a minute before coming to rest. This, like most of our other apparatus, was constructed by Mr. G. W. Thompson, the mechanician of the Jefferson Laboratory, and we have been much indebted to his skill and patience at every stage of our work. *K* is so truly hung that the outside can be used as a pulley and the whole can be rotated by the use of the belt shown in Figure 8. The vanes reach to about 2 millimeters

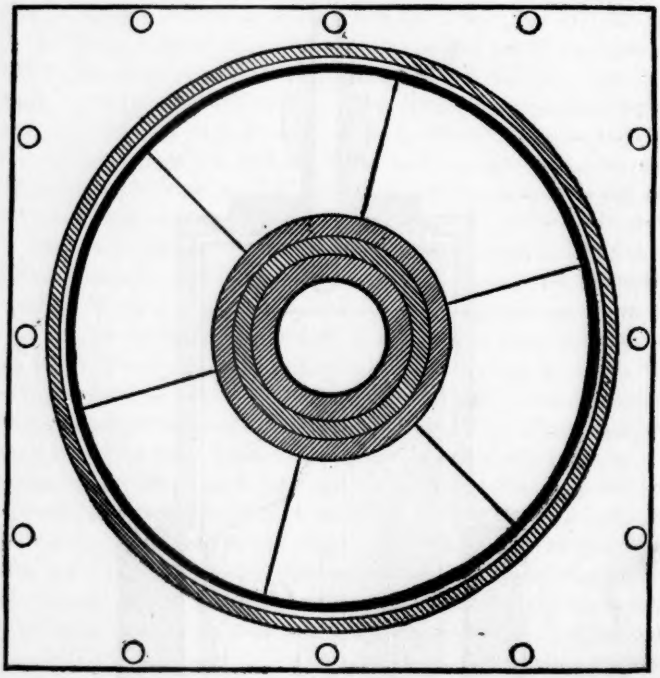

FIGURE 11.

of the floor of the box, and when the whole is filled with cracked ice and then rotated, the ice at the bottom which rubs on *Z* soon gets and holds a very smooth surface. A hole in the bottom of *Z* carries away the drip and prevents any accumulation of water on the floor of the ice box. We were at first troubled by irregularities arising from honeycombing of the ice in the ice box, and to remedy this a suitably loaded brass tripod is used to pack the ice by light blows delivered at intervals of 21 seconds, by the aid of the lever *L*. A train of four wheels is necessary to reduce the

speed of K to one revolution in 20 seconds, though only two wheels are shown in the drawings. The tripod slides in guides which revolve with K, and a swivel at the top prevents the cord from twisting.

The rotation of K and of the inside ice holder, Q, which is connected with K by a thin yoke, are matters of much importance. The continual rubbing of the ice over the flat surface of the casting seems to be necessary if the latter is to be kept at a uniform constant temperature for hours. The energy used in rotating Q is so little as to be quite negligible, as we shall show further on. The ice in K is piled up so as to cover P completely, and we have been unable to detect any difference between the temperatures within and without P by fine, properly protected thermal junctions introduced for the purpose. If, while K revolves, Q is kept still, the amount of ice melted in Q becomes irregular, though the whole amount of drip in two or three hours is not very different from the amount of steady drip in an equal time when Q is rotating. Only selected lumps of ice are put into Q. The ice to be used is first broken up into pieces weighing something like 15 grams each, by means of an ice-cracking machine, and these pieces are then put into ice water so that their sharp edges may become slightly rounded. They are then drained and dropped into Q. In this way a slight amount of water attached to the ice is introduced into Q, but the error due to this cause appears to be of slight importance. In some of our experiments the ice to be used was carefully dried in cold blotting paper, but this precaution does not seem to be necessary, though the use of small bits of ice with sharp edges is to be avoided. Q's capacity is about 2,000 cubic centimeters. After Q has been freshly filled in the course of any experiment while K is rotating, no record is kept for some time, perhaps fifteen minutes, of the amount of drip. Before the expiration of this interval the extra water introduced into Q with the ice has drained off, and the indications have become steady. After this the apparatus is allowed to run for about two hours until 300 grams of ice or less has been melted, and then Q is refilled. The drip tube always contains a few drops of water, but this amount remains sensibly constant during the progress of our experiment. The drip is collected in a graduated vessel, and the approximate amount is noted from time to time to see whether the flow is steady. The whole is then more accurately determined by weighing, at longer intervals.

The regularity of drip is a far more sensitive test of the approximate attainment of the final state of the body experimented on and its surroundings than is a sensibly constant temperature gradient on the axis.

In most of our experiments with the large apparatus just described, a sufficiently steady state has been attained in about five hours from the beginning of the heating. Sheets of blotting paper were generally inserted between the prism to be tested, and the hot and cold boxes, to serve as elastic pads, and to prevent the possible wetting of the edge of the prism by moisture condensed on the ice box. The presence of this paper prolonged the time of waiting for the final state to be attained, but did not influence the results of the measurement of the conductivity of the prism. When filled with ice, Z and K weigh about 300 kilograms, and the additional pressure due to the bolts is considerable, so that, when the prism is made up of brittle material like glass, the blotting paper or an equivalent must be used to prevent the prism from injury. We have tried several different materials, and of these the blotting paper is the most satisfactory. We may note in passing, however, that the indications of thermal couples placed between soft pads and the hard prisms are often very anomalous, two thermal junctions placed side by side sometimes differing very widely. In all the experiments that we regard as trustworthy the slab to be tested with its attendant thermopiles was placed between two other slabs *of the same material*, in forming the prism.

Most of our mercury thermometers were made by Alvergniat, or by Richards & Co., but our final standard was Tonnelot No. 11,142, upon which a very complete set of tests has been made at the International Bureau of Weights and Measures.

For temperatures higher than 100° C. we had two platinum thermometers of the general form described by Messrs. Griffiths and Callendar. These served an excellent purpose, though the wire, about 0.2 mm. in diameter, seemed from the form of the calibration curve not to be very pure. The resistance of one of them, as measured by a Carey Foster Bridge was about 29.25, 36.78, 42.85, 45.31, or 55.43 ohms, according as it was immersed in melting ice or the vapor, at 760° c.c. pressure, of water, anilin, naphthalin, or mercury. We have another thermometer made of pure platinum wire furnished by Messrs. Johnson and Matthey, 0.005 inch in diameter. This we intend to make our standard.

All our thermal elements were made either of platinoid and copper, or of German silver and copper; some were of wire, and some of narrow ribbon carefully rolled for our use. Each specimen of platinoid or German silver was "butt-jointed," generally by silver solder, to a piece

of the purest obtainable copper of equal cross section. Our finest wire thermal elements, less than one tenth of a millimeter in diameter, were so skilfully made by Mr. Sven Nelson, of Cambridge, that the joint was hardly perceptible. Our German silver and copper ribbon thermal elements, about one eighth of a millimeter thick, were made by Mr. T. W. Gleeson of Boston. These last were first soldered with the help of a holder constructed for the purpose, and the joint was then rolled or scraped until it was as nearly as might be of the same thickness as the adjacent metal.

For wire thermal elements we had large quantities of three kinds of platinoid, approximately 0.74, 0.30, and 0.097 mm. in diameter. The first two specimens were obtained about ten years ago from Messrs. Elliott Brothers, and have been thoroughly seasoned. Each is thermoelectrically pretty definite, though the two are quite different in their properties. The electromotive force, in microvolts, of platinoid and electrolytically deposited copper elements made of these wires may be tabulated as follows for low temperatures.

Temperatures of the Junctions.	Electromotive Force of Platinoid No. 1 *vs.* Copper.	Electromotive Force of Platinoid No. 2 *vs.* Copper.
0° and 10°	189	152
0° and 20°	379	306
0° and 30°	572	465
0° and 40°	769	628
0° and 50°	971	799
0° and 60°	1179	973
0° and 70°	1391	1159
0° and 80°	1609	1356
0° and 90°	1834	1569
0° and 100°	2063	1787

Besides platinoid we have used with copper for wire thermal elements two kinds of German silver wire respectively about 0.1 mm. and about

0.58 mm. in diameter. The smaller German silver wire was connected with the corresponding copper wire by a thin joint of electrolytically deposited copper. These joints were very satisfactory, but extremely tedious to make.

In some of our experiments we used fine wire thermal junctions inserted in shallow grooves accurately cut in the faces of the slabs to be tested. These grooves were made in a Brown and Sharpe Universal Milling Machine by extremely thin hard steel saws (No. 34 B. & S. Gauge) held between flat disks of somewhat smaller diameters than the saws to prevent buckling. The wire that we used fitted the grooves very closely and we hoped that the indications of the thermal couples would enable us to determine the mean temperature of the walls of the groove when the grooved slab was placed against a flat one. We soon found, however, that the results were most irregular, and, although we have spent some time in attempts to make observations obtained in this way trustworthy, we have met with little success. Sometimes our results have been good, and sometimes they have been considerably in error. We do not yet know how to make them always good. It appears that a thermal junction must be pressed firmly against a surface, the temperature of which it is to take approximately. Although we are not ready to discuss this subject exhaustively, we mention our experiences to show why we have abandoned for the present this very obvious manner of inserting thermal junctions into a prism built up out of slabs, in favor of what at first sight seems a less satisfactory device. After some preliminary experiments with fine wire thermal junctions laid between the slabs, with and without sheets of tinfoil at the sides of the wire, we determined to use the thin ribbon thermal junctions, elsewhere described, with varnished edges, so that sheets of tinfoil of the same thickness might be laid at the sides of the ribbon, and in this way a sheet of metal be introduced between the slabs.

It has been necessary for us to calibrate in the course of our work a large number of thermal elements. Some of these when properly protected we have heated with thermometers in elaborately jacketed air baths or in tanks of water or oil, and some in vapor baths. We have had considerable quantities of nearly pure chloroform, benzol, æthylen bromide, bromoform, anilin, paratoluidin, naphthalin, chinolin, α naphthol, acetanilid, naphthylamin, diphenylamin, phenanthren, anthracen, and a few other substances, the boiling points of which divide the ordinary thermometric scale below the boiling point of sulphur into small intervals. A good number of these, but not all, we have actually used.

Some of our thermopiles have been calibrated for us at temperatures between 0° C. and 100° C. by Mr. C. G. Persons of the staff of the Jefferson Laboratory, and he has assisted us in much of our other work.

The Thermal Conductivity of Marble.

With the apparatus described in this paper we have made a large number of experiments. As has been already intimated, we are not entirely satisfied with the source of heat that we have used for temperatures higher than 300° C. because of the difficulty of keeping these temperatures constant for long intervals of time, while for temperatures between 0° C. and 100° C., it has been easy to get closely agreeing results many times over. We have, nevertheless, made a good many determinations at the higher temperatures, and, while we are not yet ready to state definitely the law of variation with the temperature of the thermal conductivities of materials in which we have found such variations, we may say that, of the substances which we have examined, two, a special brand of glass of which we have a number of large plates, and dry white marble,* show no appreciable change in thermal conductivity within the limits of our measurements. We shall therefore content ourselves in this preliminary paper with giving the results of a number of determinations, made at different low temperatures, of the conductivities of about twenty specimens of marble of different kinds. Incidentally we shall need to describe very briefly some experiments upon the glass plates just mentioned.

It will appear that the conductivity of a specimen of marble at ordinary mean temperatures may depend to the amount of several per cent, as Messrs. Herschell and Lebour have shown, upon the amount of moisture which the specimen holds. For this reason we have aimed at an accuracy of only 1% in the determinations here recorded. A change in conductivity much less than this was of course easily observable. The difference of temperature between two thermopiles, one of which is only a few degrees hotter than the other, can be measured with considerable accuracy, but it will be sufficient here to state the results correct to tenths of degrees.

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* The conductivity of the specimen of marble upon which R. Weber has made a set of extremely accurate measurements appears to change by only one two-thousandth part of its own value between 0° C. and 100° C.

While it takes a long day to make an accurate determination with our large apparatus of the absolute conductivity of a slab, two determinations may easily be made in the same time of the relative conductivities of the slabs which go to form a prism, since the gradient on the axis of the slab does not sensibly change after four hours of heating, and it is then only necessary to note the readings of the thermopiles. With our smallest apparatus and thin slabs two hours are often sufficient for a measurement. Our experience seems to show that this method of comparison is susceptible of great accuracy. We have made a very large number of direct determinations of the conductivities of different slabs of stone, but, in view of the fact mentioned above that the amount of moisture in the stone affects the conductivity very appreciably, even if the less tedious method of comparison were not equally accurate, we should think it wise in future to determine with great care the absolute conductivity of a standard substance unaffected by moisture, and then compare with it the conductivity of the stone slabs. The accuracy with which the comparison can be made is greater of course than that of a single absolute determination.

The particular kind of glass which we have found useful as a comparison substance was selected some years ago from the stock of the Boston Plate Glass Company. The faces of each plate are very nearly plane, but the planes are not in every specimen quite parallel. The conductivities of different plates are somewhat different, but the conductivity of each plate remains sensibly constant within large ranges of temperature. Cut from this glass we have a number of slabs 60 centimeters square, a number of slabs 30 centimeters square and some disks about 20 centimeters in diameter.

We shall wish to discuss the properties of this kind of glass at higher temperatures more particularly on another occasion. For our present purposes, it is worth while to measure the temperatures to tenths of degrees only and the thickness of a slab to the nearest twentieth of a millimeter, and an account of a few experiments to this degree of accuracy, chosen almost at random from the large number of which we have records, will suffice.

Slabs A, B, C, and D are cut from one particular large homogeneous piece of this glass, the conductivity of which, according to our determinations, is to that of Plate III. mentioned below as 187 to 175. We shall assume the conductivities of these slabs to be 0.00277 at all ordinary temperatures. We have not been able to detect any differences in their conductivities.

Experiment (*a*). A compound slab, made up of slabs B and A with their thermal elements, was placed between two other glass plates to form a prism. The thickness of B is 0.950 cm. and of A 0.935 cm. In the final state of the prism, the thermal elements on the warmer face of B, between B and A, and on the colder side of A, indicated 88°.1, 63°.4, and 38°.9 respectively. A fall of 14°.7 in 0.950 cm. is very nearly equal to a fall of 14°.5 in 0.935 cm.

Experiment (*b*). In the final state of a prism made up of slabs A and B shut in between two other glass plates, the thermal elements on the warmer face of A, between A and B, and on the colder face of B, indicated 85°.0, 62°.2, and 39°.1 respectively. A fall of 22°.8 in 0.935 cm. is very nearly equal to a fall of 23°.1 in 0.950 cm.

Experiment (*c*). Three slabs A, C, and E of the standard glass with three other glass plates, which we may denote by P, Q, and R, were built up into a prism $P\,A\,Q\,C\,E\,R$ with thermal elements between P and A, A and Q, Q and C, E and R. In the final state the temperatures of the thermal elements were very nearly 88°.2, 74°.2, 58°.8 and 30°.0, respectively, so that the gradient in the slab A of thickness 0.935 cm. is almost exactly the same as in the double slab $C\,E$ of thickness 1.93 cm. There seemed to be, therefore, no appreciable contact resistance (Uebergangswiderstand) between the two slabs.

Experiment (*d*). After experiment (*c*) had been finished, a narrow ring of blotting paper, the inside diameter of which was only slightly less than the diameter of the disks, was inserted between C and E so as to have a dead air space between them 0.7 mm. thick, when the prism was under pressure. In the final state the temperatures were now 89°.9, 78°.3, 66°.5, and 25°.9, so that in this particular case the dead air space was nearly equivalent to a glass plate 4.8 mm. thick.

Experiment (*e*). In this experiment Plate III., of 0.875 cm. thickness, was a part of a prism heated in the larger apparatus intended for the determination of absolute conductivities. The temperatures of the thermal elements on the faces of the plates in the final state were 69°.7 and 58°.8 respectively. In 9060 seconds 464.5 grams of ice were melted. Assuming the area of the bottom of the ice pot to be 126.7 square centimeters and the latent heat of melting of ice to be 79.25, this corresponds to a conductivity of 0.00258. It is obvious, however, that the last figure of this number is not quite definitely determined.

Experiment (*f*). In the final state of a prism similar to the one used in the last experiment, 311.9 grams of ice were melted in 5340 seconds when the temperatures of the thermal elements on the faces of Plate III. were 66°.4 and 54°.1. This corresponds to a conductivity of 0.00260. Here again the last figure is in doubt.

We had occasion to measure the absolute conductivity of only one other of the 60 cm. square plates bought at the same time as Plate III. This was Plate I. The results of two experiments made on it were 0.00262 and 0.00259. The crown glass used by Oddone had a conductivity of 0.00245, that of Lees * a conductivity of 0.00243.

We will next cite a single experiment to show how much the conductivity of the particular kind of statuary marble that we used could be changed by moistening the stone.

Experiment (*g*). A prism was made up of three plates of glass, *A*, *P*, and *Q*, and three dry slabs of statuary marble, *C*, *D*, and *E*, arranged in the order *P A Q E D C* with thermal junctions between *P* and *A*, *A* and *Q*, *E* and *D*, *D* and *C*. The temperatures indicated by the thermal junctions when the prism had sensibly reached its final state were 84°.6, 67°.7, 38°.6, and 27°.7. *D* was then well moistened with water, and the experiment was then repeated. The temperatures were then 85°.3, 70°.5, 46°.0, and 38°.1, so that the conductivity of *D* had been increased in the ratio of 1.21 to 1.

Experiment (*h*). In order to form an idea of the amount of change with the state of the weather of the conductivity of a piece of our Carrara statuary marble, we made three comparisons on three different occasions of the relative conductivities of a slab of it (*C*) 1.08 centimeters thick, and a plate (*A*) of standard glass. Between the experiments, *C* was left in a room the windows of which were much of the time open. The results were as follows: —

Temperature of the warm side of the glass,	85°.5	84°.6	84°.8
Temperature of the cool side of the glass,	68°.1	67°.1	67°.4
Temperature of the warm side of the marble,	43°.0	42°.1	40°.3
Temperature of the cool side of the marble,	32°.0	31°.1	29°.4
Ratio of the conductivities of the marble and the glass,	1.84	1.84	1.83
Average conductivity of the slab *C*,		0.00509	

Another specimen of Carrara marble had a conductivity of 0.00501.

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* We have not yet seen the paper by Mr. Lees mentioned in the March, 1898, number of the Beiblätter zu den Annalen der Physik und Chemie.

Before we state the results of our own observations upon other specimens, we will give for purposes of comparison some determinations of the thermal conductivities of marble made by other observers.

Material.	Observers.	Conductivity.
"Carrara Marble"	Stadler	0.0049
"White Italian Veined Marble"	Herschel, Lebour, and Dunn	0.0051
"Irish Green Marble"	" "	0.0051
"Irish Fossil Marble"	" "	0.0052
"White Sicilian Marble"	" "	0.0054
White Marble	R. Weber	$0.0054(1.-0.000005t)$
White Marble	Lees	0.0071
"White Coarse-grained Marble"	Yamagawa	0.0073
"Sugar White Coarse-grained Marble"	Péclet	0.0077
"Fine-grained Gray Marble"	Péclet	0.0097

Taking a certain piece of "Pyrenees Marble" as a standard, Dr. Less found the conductivities of specimens of "Carrara Marble" and "Italian Marble" to be 0.769 and 0.763 respectively.

In determining the thermal conductivities of the specimens of marble mentioned below, the prism clamped between the hot and the cold box of our apparatus was made up of six slabs in series, a plate of standard glass 0.935 cm. thick between two thin plates of glass, and the slab to be tested between two thin slabs of marble. A ribbon thermal element and tinfoil wings were placed on each side of the standard glass, and on each side of the marble to be experimented on, so that there were four of these thermal elements in all. When the prism had sensibly reached its final state, the temperatures of the thermal junctions were determined and the ratio of the conductivities of the glass and the marble was assumed to be equal to the reciprocal of the ratio of the gradients in the two slabs. By introducing an extra plate or a sheet or two of blotting paper into the prism, the two gradients could be altered at pleasure but not their ratio. So far as we could see, it was immaterial in the case of these substances whether the marble base of the prism or the glass base was placed uppermost, but we generally placed the marble on top, so that the mean temperature of each specimen might be about 30° C. In stating the results of some of these determinations, we shall give the temperatures of the four thermal junctions in order, then the ratio of the conductivities of the marble to be tested and the standard glass, and finally the absolute conductivity of the marble on the assumption that that of the glass is 0.00277. We shall give the absolute conductivity of the marble to three significant figures, but it is evident that the last of these is not determined. All the specimens were artificially dried for some time in the hot air space over the boilers which furnish steam for heating the Jefferson Laboratory, and were then allowed to stand for some weeks at ordinary room temperatures so that their conditions might be normal. The artificial heating drove off the excess of moisture acquired by the marble while being cut under water at the mill.

Most of our stone was obtained from Messrs. Bowker and Torrey of Boston, who kindly collected for us representative specimens of such materials as are commonly used for decorative and monumental purposes. We have given to the slabs the names used by stone workers and have called them all "marbles," though one or two might more properly be called "limestones." The "Mexican Onyx" is really travertine. Our thanks are due to Prof. J. E. Wolff for help in identifying our specimens.

Fossiliferous Tennessee Marble.

(Red with numerous white fossils.)

Thickness in centimeters,	2.40
Temperatures of the faces of the glass plate,	82°.3 and 63°.2
Temperatures of the faces of the marble slab,	43°.3 and 24°.4
Ratio of the conductivities of the marble and the glass,	2.73
Absolute conductivity of the marble,	0.00756

American White Marble.

(Cream white.)

Thickness in centimeters,	2.68
Temperatures of the faces of the glass plate,	83°.6 and 64°.6
Temperatures of the faces of the marble slab,	45°.4 and 20°.3
Ratio of the conductivities of the marble and the glass,	2.15
Absolute conductivity of the marble,	0.00596

Vermont Statuary Marble.

(Snow white with coarse but uniform grain.)

Thickness in centimeters,	2.40
Temperatures of the faces of the glass plate,	82°.9 and 64°.2
Temperatures of the faces of the marble slab,	44°.7 and 21°.7
Ratio of the conductivities of the marble and the glass,	2.09
Absolute conductivity of the marble,	0.00578

Lisbon Marble.

(Light terra-cotta with darker veins.)

Thickness in centimeters,	2.30
Temperatures of the faces of the glass plate,	80°.9 and 60°.8
Temperatures of the faces of the marble slab,	39°.6 and 19°.6
Ratio of the conductivities of the marble and the glass,	2.47
Absolute conductivity of the marble	0.00685

St. Baume Marble.

(Yellow, red, and yellowish white brecciated.)

Thickness in centimeters,	2.36
Temperatures of the faces of the glass plate,	80°.9 and 61°.2
Temperatures of the faces of the marble slab,	40°.3 and 22°.1
Ratio of the conductivities of the marble and the glass,	2.75
Absolute conductivity of the marble,	0.00761

Rose Ivory Marble.

(From Djebel-er-Roos, Algiers. White with very slight pinkish tinge. Very fine in grain.)

Thickness in centimeters,	2.64
Temperatures of the faces of the glass plate,	80°.3 and 60°.2
Temperatures of the faces of the marble slab,	39°.8 and 19°.0
Ratio of the conductivities of the marble and the glass,	2.73
Absolute conductivity of the marble,	0.00756

Italian Egyptian Marble.

(Breccia. Slate colored with ochre-yellow and white veins.)

Thickness in centimeters,	2.55
Temperatures of the faces of the glass plate,	83°.0 and 63°.3
Temperatures of the faces of the marble slab,	43°.1 and 19°.2
Ratio of the conductivities of the marble and the glass,	2.25
Absolute conductivity of the marble,	0.00623

Mexican Onyx.

(Alabaster white, translucent.)

Thickness in centimetres,	2.29
Temperatures of the faces of the glass plate,	82°.9 and 63°.8
Temperatures of the faces of the onyx slab,	43°.1 and 19°.8
Ratio of the conductivities of the onyx and the glass,	2.01
Absolute conductivity of the oynx,	0.00556

Vermont Dove Colored Marble.

(Dove colored with light and dark striæ.)

Thickness in centimeters,	2.19
Temperatures of the faces of the glass plate,	80°.5 and 59°.3
Temperatures of the faces of the marble slab,	39°.1 and 18°.9
Ratio of the conductivities of the marble and the glass,	2.47
Absolute conductivity of the marble,	0.00684

Bardiglio Marble.

(From the Seravazza quarries. Cloudy white, with network of distinct dark lines.)

Thickness in centimeters,	2.44
Temperatures of the faces of the glass plate,	81°.8 and 61°.3
Temperatures of the faces of the marble plate,	41°.1 and 19°.3
Ratio of the conductivities of the marble and the glass,	2.45
Absolute conductivity of the marble,	0.00680

Sienna Marble.

(Yellowish white with blue veins.)

Thickness in centimeters,	2.48
Temperatures of the faces of the glass plate,	81°.5 and 60°.9
Temperatures of the faces of the marble plate,	40°.9 and 18°.6
Ratio of the conductivities of the marble and the glass,	2.44
Absolute conductivity of the marble,	0.00676

St. Anne Marble.

(Brown black with white patches.)

Thickness in centimeters,	2.34
Temperatures of the faces of the glass plate,	80°.9 and 60°.1
Temperatures of the faces of the marble plate,	38°.8 and 19°.7
Ratio of the conductivities of the marble and the glass,	2.73
Absolute conductivity of the marble,	0.00755

American Black Marble.

(Dark slate.)

Thickness in centimeters,	2.43
Temperatures of the faces of the glass plate,	81°.0 and 61°.1
Temperatures of the faces of the marble slab,	40°.1 and 19°.2
Ratio of the conductivities of the marble and the glass,	2.47
Absolute conductivity of the marble,	0.00685

Vermont Cloudy Marble.

(Cloudy white with darker patches.)

Thickness in centimeters,	2.55
Temperatures of the faces of the glass plate,	82°.3 and 62°.1
Temperatures of the faces of the marble slab,	41°.8 and 19°.4
Ratio of the conductivities of the marble and the glass,	2.46
Absolute conductivity of the marble,	0.00681

Knoxville Marble

(Pink with occasional dark serrated veins.)

Thickness in centimeters,	2.37
Temperatures of the faces of the glass plate,	81°.6 and 61°.0
Temperatures of the faces of the marble slab,	38°.9 and 20°.1
Ratio of the conductivities of the marble and the glass,	2.62
Absolute conductivity of the marble,	0.00757

Arranging the results in the order of the conductivities of the specimens, we get the subjoined table. We call attention to the two groups of fine-grained marbles, which have conductivities of about 0.0068 and 0.0076 respectively, at about 30° C.

Variety of Marble.	Conductivity.
"Carrara Statuary"	0.00501
" "	0.00509
"Mexican Onyx"	0.00556
"Vermont Statuary"	0.00578
"American White"	0.00596
"Egyptian"	0.00623
"Sienna"	0.00676
"Bardiglio"	0.00680
"Vermont Cloudy White"	0.00681
"Vermont Dove Colored"	0.00684
"Lisbon"	0.00685
"American Black"	0.00685
"Belgian"	0.00755
"African Rose Ivory"	0.00756
"Tennessee Fossiliferous"	0.00756
"Knoxville Pink"	0.00757
"St. Baume"	0.00761

We reserve for a second paper the results of observations made upon other materials.

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JEFFERSON PHYSICAL LABORATORY,
CAMBRIDGE, MASS.